

# Optimal Control of Two-Player Systems with Output Feedback

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## Abstract

In this article, we consider a fundamental decentralized optimal control problem, which we call the two-player problem. Two subsystems are interconnected in a nested information pattern, and output feedback controllers must be designed for each subsystem. Several special cases of this architecture have previously been solved, such as the state-feedback case or the case where the dynamics of both systems are decoupled. In this paper, we present a detailed solution to the general case. The structure of the optimal decentralized controller is reminiscent of that of the optimal centralized controller; each player must estimate the state of the system given their available information and apply static control policies to these estimates to compute the optimal controller. The previously solved cases benefit from a separation between estimation and control which allows one to compute the control and estimation gains separately. This feature is not present in general, and some of the gains must be solved for simultaneously. We show that computing the required coupled estimation and control gains amounts to solving a small system of linear equations.

## I Introduction

Many large-scale systems such as the internet, power grids, or teams of autonomous vehicles, can be viewed as a network of interconnected subsystems. A common feature of these applications is that subsystems must make control decisions with limited information. The hope is that despite the decentralized nature of the system, global performance criteria can be optimized.

In this paper, we consider a particular class of two-player decentralized control problems, illustrated in Figure 1, and develop the optimal controller in this framework. In recent years, much work has gone into developing results which characterize exactly when the standard linear-quadratic optimal control problem has a linear optimal controller. The problem we consider fulfils these criteria, but beyond the fact that the optimal controller is linear, very little has been hitherto known about its structure. This particular configuration is therefore chosen with some care, to be the simplest possible directed graph with measurement feedback that admits a linear optimal controller.

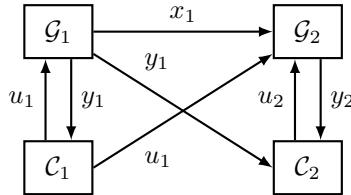


Figure 1: Decentralized interconnection

Figure 1 shows two plants  $\mathcal{G}_1$  and  $\mathcal{G}_2$ , with associated controllers  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . Controller  $\mathcal{C}_1$  receives information only from plant  $\mathcal{G}_1$ , whereas  $\mathcal{C}_2$  receives information from both  $\mathcal{G}_1$  and  $\mathcal{G}_2$ . The actions of  $\mathcal{C}_1$  affect both  $\mathcal{G}_1$  and  $\mathcal{G}_2$ , whereas the actions of  $\mathcal{C}_2$  only affect  $\mathcal{G}_2$ . Furthermore,  $\mathcal{G}_1$  affects  $\mathcal{G}_2$ , but not vice-versa. In other words, information only flows from left-to-right. This is the simplest nontrivial structure where linear controllers are optimal for linear plants. This feedback interconnection may be represented using the linear fractional transformation, as shown in Figure 2.

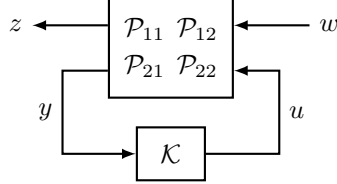


Figure 2: General four-block plant with controller in feedback

In doing so, the graph structure results in a sparsity structure for both  $\mathcal{P}$  and  $\mathcal{K}$ . Specifically, we find the following block-lower-triangular sparsity pattern

$$\mathcal{P}_{22} = \begin{bmatrix} \mathcal{G}_{11} & 0 \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{bmatrix} \quad \text{and} \quad \mathcal{K} = \begin{bmatrix} \mathcal{K}_{11} & 0 \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix}$$

while  $\mathcal{P}_{11}$ ,  $\mathcal{P}_{12}$ , and  $\mathcal{P}_{21}$  are full in general. We assume the plant and controller are finite-dimensional continuous-time linear time-invariant systems. The goal is to find an  $\mathcal{H}_2$ -optimal controller  $\mathcal{K}$  subject to this structural constraint. We paraphrase our main result, found in Theorem 11. Consider the state-space dynamics

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + w \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v \end{aligned}$$

where  $w$  and  $v$  are white noise. The objective is to minimize the quadratic cost of the standard LQG problem, subject to the constraint that

Player 1 measures  $y_1$  and chooses  $u_1$   
 Player 2 measures  $y_1, y_2$  and chooses  $u_2$

We show that the optimal controller is

$$u = Kx_{|y_1, u_\zeta} + \begin{bmatrix} 0 & 0 \\ H & J \end{bmatrix} (x_{|y, u} - x_{|y_1, u_\zeta})$$

where  $x_{|y_1, u_\zeta}$  denotes the optimal estimate of  $x$  using the information available to Player 1, and  $x_{|y, u}$  is the optimal estimate of  $x$  using the information available Player 2. The matrices  $K$ ,  $J$ , and  $H$  are determined from the solutions to Riccati and Sylvester equations. The precise meaning of *optimal estimate* and  $u_\zeta$  will be explained in Section V. Our results therefore provide a type of separation principle for such problems. Questions of separation are central to decentralized control, and very little is known in the general case. Our results therefore also shed some light on this important issue. Even though the controller is expressed in terms of optimal estimators, these do not all evolve according to a standard Kalman filter, since Player 1, for example, does not know  $u_2$ .

The main contribution of this paper is an explicit state-space formula for the optimal controller, which was not previously known. The realization we find is generally minimal, and computing it is of comparable computational complexity to computing the optimal centralized controller. The solution also gives an intuitive interpretation for the states of the optimal controller. The analysis of stabilization of such structured systems is also a significant contribution of this paper. We give a state-space parameterization of all stabilizing decentralized controllers for this problem.

The paper is organized as follows. In the remainder of the introduction, we give a brief history of decentralized control and the two-player problem in particular. Then, we cover some background mathematics and give a formal statement of the two-player optimal control problem. In Section III, we characterize all structured stabilizing controllers, and show that the two-player control problem can be expressed as an equivalent structured model-matching problem, which is convex. In Section IV we state and prove our main result which contains the optimal controller formulae. We follow up with a discussion of the structure and interpretation of this controller in Section V. The subsequent Section VI gives an explicit construction of the controller. Finally, we conclude in Section VII.

## I-A Prior work

If we consider the problem of Section I but remove the structural constraint on the controller, the problem becomes the well-studied classical  $\mathcal{H}_2$  synthesis, solved for example in [30]. The optimal controller in this case is linear, and has as many states as the original system.

The presence of structural constraints greatly complicates the problem, and the resulting decentralized problem has been outstanding since the seminal paper by Witsenhausen [26] in 1968. Witsenhausen posed a related problem for which a nonlinear controller strictly outperforms all linear policies. Not all structural constraints lead to nonlinear optimal controllers, however. For a broad class of decentralized control problems there exists a linear optimal policy, and finding it amounts to solving a convex optimization problem [5, 15, 17, 24]. The two-player problem is in this class.

Despite the benefit of convexity, the search space is infinite-dimensional since we must optimize over transfer functions. Several numerical and analytical approaches for addressing decentralized optimal control exist, including [15, 16, 20, 29]. One particularly relevant numerical approach is to use vectorization, which converts the decentralized problem into an equivalent centralized problem [18]. This conversion process results in a dramatic growth in state dimension, and so the method is extremely computationally intensive and only feasible for small problems. However, it does provide important insight into the problem. Namely, it proves that there exists an optimal rational controller for the two-player problem considered herein, and gives an upper bound on the state dimension. These results are presented in Lemma 13.

A drawback of such numerical approaches is that they do not provide an intuitive explanation for what the controller is doing; there is no physical interpretation for the states of the controller. In the centralized case, we have such an interpretation. Specifically, the controller consists of a Kalman filter whose states are estimates of the states of the plant, and a static control gain that corresponds to the solution of an LQR problem. Recent structural results [1, 14] show that for general classes of delayed-sharing information patterns, the optimal control policy depends on a summary of the information available and a dynamic programming approach may be used to compute it. For linear systems in which each player eventually has access to all information, explicit formulae were computed in [8]. There are general results also in the case where not all information is eventually available to all players [27]. However, these results do not appear to easily translate to state-space formulae for linear systems.

Certain special cases of the two-player problem have been solved explicitly and clean physical interpretations have been found for the states of the optimal controller. Most notably, the state-feedback case admits an explicit state-space solution using a spectral factorization approach [22]. This approach was also used to address a case with partial output-feedback, in which there is output-feedback for one player and state-feedback for the other [23]. The work of [21] also provided a solution to the state-feedback case using the Möbius transform associated with the underlying poset. Certain special cases were also solved in [7], which gave a method for splitting decentralized optimal control problems into multiple centralized problems. This splitting approach addresses problems other than state-feedback, including partial output-feedback, and dynamically decoupled problems.

In this article, we address the general two-player output-feedback problem. Our approach is perhaps closest technically to the work of [22] using spectral factorization, but uses the factorization to split the problem in a different way, allowing a solution of the general output-feedback problem. We also provide a meaningful interpretation of the states of the optimal controller. This paper is a substantially more general version of the invited paper [11] and the conference paper [12], where the special case of stable systems was considered. We also mention the related work [9] which addresses star-shaped systems in the stable case.

## II Preliminaries

We use  $\mathbb{Z}_+$  to denote the nonnegative integers. The imaginary unit is  $j$ , and we denote the imaginary axis by  $j\mathbb{R}$ . A square matrix  $A \in \mathbb{R}^{n \times n}$  is Hurwitz if all of its eigenvalues have a strictly negative real part. The set  $\mathcal{L}_2(j\mathbb{R})$ , or simply  $\mathcal{L}_2$ , is a Hilbert space of Lebesgue measurable matrix-valued functions  $\mathcal{F} : j\mathbb{R} \mapsto \mathbb{C}^{m \times n}$  with the inner product

$$\langle \mathcal{F}, \mathcal{G} \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}(\mathcal{F}^*(j\omega)\mathcal{G}(j\omega)) d\omega$$

such that the inner product induced norm  $\|\mathcal{F}\|_2 = \langle \mathcal{F}, \mathcal{F} \rangle^{1/2}$  is bounded. We will sometimes write  $\mathcal{L}_2^{m \times n}$  to be explicit about the matrix dimensions. As is standard,  $\mathcal{H}_2$  is a closed subspace of  $\mathcal{L}_2$  with matrix functions analytic in the open right-half plane.  $\mathcal{H}_2^\perp$  is the orthogonal complement of  $\mathcal{H}_2$  in  $\mathcal{L}_2$ . We write  $\mathcal{R}_p$  to denote the set of proper real rational transfer functions. We also use  $\mathcal{R}$  as a prefix to modify other sets to indicate the restriction to real rational functions. So  $\mathcal{R}\mathcal{L}_2$  is the set of strictly proper rational transfer functions with no poles on the imaginary axis, and  $\mathcal{R}\mathcal{H}_2$  is the stable subspace of  $\mathcal{R}\mathcal{L}_2$ . The set of stable proper transfer functions is denoted  $\mathcal{RH}_\infty$ . Every  $\mathcal{G} \in \mathcal{R}_p$  has a state-space realization

$$\mathcal{G} = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = D + C(sI - A)^{-1}B \quad \text{with} \quad \mathcal{G}^* = \left[ \begin{array}{c|c} -A^\top & C^\top \\ \hline -B^\top & D^\top \end{array} \right]$$

where  $\mathcal{G}^*$  is the conjugate transpose of  $\mathcal{G}$ . If this realization is chosen to be stabilizable and detectable, then  $\mathcal{G} \in \mathcal{RH}_\infty$  if and only if  $A$  is Hurwitz, and  $\mathcal{G} \in \mathcal{RH}_2$  if and only if  $A$  is Hurwitz and  $D = 0$ . If  $z = \mathcal{G}w$  where  $\mathcal{G} \in \mathcal{RH}_2$  and  $w$  is white Gaussian noise with unit variance, we have the following well-known relationship between the norm of the signal  $z$  and the  $\mathcal{L}_2$ -norm of the system  $\mathcal{G}$ .

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \int_0^T \|z(t)\|^2 dt = \|\mathcal{G}\|_2^2$$

For the remainder of this paper, whenever we write  $\|\mathcal{G}\|_2$ , it will always be the case that  $\mathcal{G} \in \mathcal{RH}_2$ . For a thorough introduction to these topics, see [30].

The plant  $\mathcal{P} \in \mathcal{R}_p$  maps exogenous inputs  $w$  and actuator inputs  $u$  to regulated outputs  $z$  and measurements  $y$ . Our objective is to find a control law  $u = \mathcal{K}y$  where  $\mathcal{K} \in \mathcal{R}_p$  so that the closed-loop map has some desirable properties. The closed-loop map is depicted in Figure 2, and this figure corresponds to

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \quad (1)$$

$$u = \mathcal{K}y$$

The closed-loop map from  $w$  to  $z$  is given by the lower linear fractional transform (LFT) defined by  $F_\ell(\mathcal{P}, \mathcal{K}) = \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21}$ . If  $\mathcal{K} = F_\ell(\mathcal{J}, \mathcal{Q})$  and  $\mathcal{J}$  has a proper inverse, the LFT may be inverted according to  $\mathcal{Q} = F_u(\mathcal{J}^{-1}, \mathcal{K})$ , where  $F_u$  denotes the upper LFT, defined as  $F_u(\mathcal{M}, \mathcal{K}) = \mathcal{M}_{22} + \mathcal{M}_{21}\mathcal{K}(I - \mathcal{M}_{11}\mathcal{K})^{-1}\mathcal{M}_{12}$ .

We will express the results of this paper in terms of the solutions to algebraic Riccati equations (ARE) and recall here the basic facts. If  $D^\top D > 0$ , then the following are equivalent.

- (i) There exists  $X \in \mathbb{R}^{n \times n}$  such that

$$A^\top X + XA + C^\top C - (XB + C^\top D)(D^\top D)^{-1}(B^\top X + D^\top C) = 0 \quad (2)$$

and  $A - B(D^\top D)^{-1}(B^\top X + D^\top C)$  is Hurwitz

- (ii)  $(A, B)$  is stabilizable and  $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$  has full column rank for all  $\omega \in \mathbb{R}$ .

Under these conditions, there is a unique  $X \in \mathbb{R}^{n \times n}$  satisfying (i). This  $X$  is symmetric and positive semidefinite, and is called the *stabilizing solution* of the algebraic Riccati equation. As a short form we will write  $(X, K) = \text{ARE}(A, B, C, D)$  where  $K = -(D^\top D)^{-1}(B^\top X + D^\top C)$  is the associated gain.

## II-A Block-triangular matrices

Due to the triangular structure of the problem considered herein, we also require notation to denote the sets of block-lower-triangular matrices. To this end, suppose  $S$  is a commutative ring,  $m, n \in \mathbb{Z}_+^2$  and  $m_i, n_i \geq 0$ . We define  $\text{lower}(S, m, n)$  to be the set of block-lower-triangular matrices with elements in  $S$  partitioned according to the index sets  $m$  and  $n$ . That is,  $X \in \text{lower}(S, m, n)$  if and only if

$$X = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix} \quad \text{where} \quad X_{ij} \in S^{m_i \times n_j}$$

We sometimes omit the indices and simply write  $\text{lower}(S)$ . We also define the matrices  $E_1^\top = [I \ 0]^\top$  and  $E_2^\top = [0 \ I]^\top$ , with dimensions inferred from context. For example, if we write  $XE_1$  where  $X \in \text{lower}(\mathbb{R}, m, n)$ , then  $E_1 \in \mathbb{R}^{(n_1+n_2) \times n_1}$ . When writing  $A \in \text{lower}(S)$ , we allow for the possibility that some of the blocks may be empty. For example, if  $m_1 = 0$  then we encounter the trivial case where  $\text{lower}(S, m, n) = S^{m_2 \times (n_1+n_2)}$ .

There is a correspondence between proper transfer functions  $\mathcal{G} \in \mathcal{R}_p$  and state-space realizations  $(A, B, C, D)$ . The next result shows that a specialized form of this correspondence exists when  $\mathcal{G} \in \text{lower}(\mathcal{R}_p)$ .

**Theorem 1.** Suppose  $\mathcal{G} \in \text{lower}(\mathcal{R}_p, k, m)$ , and a realization for  $\mathcal{G}$  is given by  $(A, B, C, D)$ . Then there exists  $n \in \mathbb{Z}_+^2$  and an invertible matrix  $T$  such that

$$\begin{aligned} T^{-1}AT &\in \text{lower}(\mathbb{R}, n, n) & T^{-1}B &\in \text{lower}(\mathbb{R}, n, m) \\ CT &\in \text{lower}(\mathbb{R}, k, n) & D &\in \text{lower}(\mathbb{R}, k, m) \end{aligned}$$

**Proof.** Partition the state-space matrices according to the partition imposed by  $k$  and  $m$ .

$$\mathcal{G} = \begin{bmatrix} \mathcal{G}_{11} & 0 \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{bmatrix} = \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & 0 \\ C_2 & D_{21} & D_{22} \end{array} \right]$$

Note that we immediately have  $D \in \text{lower}(\mathbb{R}, k, m)$ . However,  $A$ ,  $B$ , and  $C$  need not have the desired structure. If  $\mathcal{G}_{12}$  is empty, the sparsity pattern is trivial and any realization  $(A, B, C, D)$  will do. Suppose  $\mathcal{G}_{12}$  is non-empty and let  $T$  be the matrix that transforms it into Kalman canonical form.

There are typically four blocks in such a decomposition, but since  $\mathcal{G}_{12} = 0$ , there can be no modes that are both controllable and observable. Apply the same  $T$ -transformation to  $\mathcal{G}$ , and obtain the realization

$$\mathcal{G} = \left[ \begin{array}{ccc|cc} A_{\bar{c}o} & 0 & 0 & B_{11} & 0 \\ A_{21} & A_{\bar{c}o} & 0 & B_{21} & 0 \\ A_{31} & A_{32} & A_{c\bar{o}} & B_{31} & B_{c\bar{o}} \\ \hline C_{\bar{c}o} & 0 & 0 & D_{11} & 0 \\ C_{21} & C_{22} & C_{23} & D_{21} & D_{22} \end{array} \right] \quad (3)$$

This realization has the desired sparsity pattern, and we notice that there may be many admissible index sets  $n$ . For example, the modes  $A_{\bar{c}o}$  can be included into either the  $A_{11}$  block or the  $A_{22}$  block. These modes correspond to the modes of  $\mathcal{G}_{21}$  that are not in common with the modes of  $\mathcal{G}_{11}$  or  $\mathcal{G}_{22}$ . Also note that (3) is an admissible realization even if some of the diagonal blocks of  $A$  in this realization are empty. ■

## II-B Stabilization

We seek controllers  $\mathcal{K} \in \mathcal{R}_p$  such that when they are connected in feedback to a plant  $\mathcal{P} \in \mathcal{R}_p$  as in Figure 2, the plant is stabilized. Consider the interconnection in Figure 3. We say that  $\mathcal{K}$  stabilizes  $\mathcal{P}$  if the transfer function  $(w, u_1, u_2) \mapsto (z, y_1, y_2)$  is well-posed and stable. Well-posedness amounts to  $I - \mathcal{P}_{22}(\infty)\mathcal{K}(\infty)$  being nonsingular. Similarly, if we consider the interconnection in Figure 4, we say that  $\mathcal{K}$  stabilizes  $\mathcal{P}_{22}$  if the transfer function  $(u_1, u_2) \mapsto (y_1, y_2)$  is well-posed and stable. The well-posedness condition is the same in this case as in the case of stabilizing  $\mathcal{P}$ . The two notions of stabilization are discussed in [2–4, 30] and are related via the following result. Stabilization can also be characterized using state-space, which we give in Proposition 3.

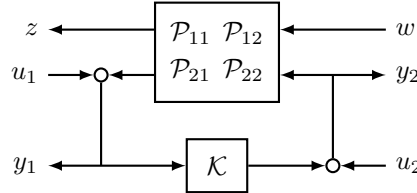


Figure 3: Feedback loop with additional inputs and outputs for analysis of stabilization of two-input two-output systems

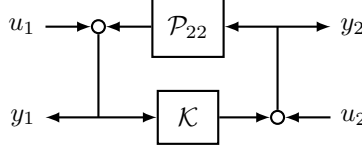


Figure 4: Feedback loop with additional inputs and outputs for analysis of stabilization

**Proposition 2** (see [4, Lemma A.4.3]). *Suppose there exists a  $K_0 \in \mathcal{R}_p$  such that  $K_0$  stabilizes  $\mathcal{P}$ . Then for any  $K \in \mathcal{R}_p$ ,  $K$  stabilizes  $\mathcal{P}$  if and only if  $K$  stabilizes  $\mathcal{P}_{22}$ .*

### II-C Problem statement

The problem addressed in this article may be formally stated as follows. Suppose  $\mathcal{P} \in \mathcal{R}_p$  is given. Further suppose that  $\mathcal{P}_{22} \in \text{lower}(\mathcal{R}_p, k, m)$ . The two-player problem is

$$\begin{aligned} & \text{minimize} && \|\mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21}\|_2 \\ & \text{subject to} && \mathcal{K} \in \text{lower}(\mathcal{R}_p, m, k) \\ & && \mathcal{K} \text{ stabilizes } \mathcal{P} \end{aligned} \tag{4}$$

We will also make some additional assumptions. We will assume without loss of generality that  $\mathcal{P}_{11}$  and  $\mathcal{P}_{22}$  are strictly proper and we will make some standard technical assumptions on  $\mathcal{P}_{12}$  and  $\mathcal{P}_{21}$  in order to guarantee the existence and uniqueness of the optimal controller. The first step in our solution to the two-player problem (4) is to deal with the stabilization constraint. This is the topic of the next section.

## III Stabilization of triangular systems

In this section, we provide a state-space characterization of stabilization when both the plant and controller have a block-lower-triangular structure. Specifically, we find necessary and sufficient conditions under which a block-lower-triangular stabilizing controller exists, and we provide a parameterization of all such controllers akin to the Youla parameterization [28].

The fundamental results regarding decentralized stabilizability of linear systems were developed in [25] via analysis of the system transfer functions. We present a state-space development of the triangular case, aimed at parameterizing all stabilizing controllers. We consider both the state-space and input-output characterizations of stabilization. Our approach closely parallels that in [30] for centralized systems, and we will use their notation whenever possible. Some of these cases have been addressed in [15] using a coprime factorization approach. Related work has also appeared in [19]. Many of the well-known equivalences between these two forms of stability are much more subtle in the decentralized case, and in this section we therefore carefully delineate these issues.

Throughout this section, we assume the plant  $\mathcal{P}$  has a minimal realization that is partitioned according to the partition imposed by the  $\mathcal{P}_{ij}$ .

$$\begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} = \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \tag{5}$$

When dealing with the two-player problem, we further assume that the  $\mathcal{P}_{22}$  subsystem is block-lower-triangular. Since we have  $\mathcal{P}_{22} \in \text{lower}(\mathcal{R}_p)$ , Theorem 1 implies that there

exists a minimal realization of  $\mathcal{P}$  for which

$$A \triangleq \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad B_2 \triangleq \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \quad C_2 \triangleq \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix} \quad D_{22} \triangleq \begin{bmatrix} \bar{D}_{11} & 0 \\ \bar{D}_{21} & \bar{D}_{22} \end{bmatrix} \quad (6)$$

and we therefore assume that  $A, B_2, C_2, D_{22}$  have this form. Note that the block sizes must be consistent, so there exists  $n, m, k \in \mathbb{Z}_+^2$  such that  $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ ,  $B_{ij} \in \mathbb{R}^{n_i \times m_j}$ ,  $C_{ij} \in \mathbb{R}^{k_i \times n_j}$ , and  $\bar{D}_{ij} \in \mathbb{R}^{k_i \times m_j}$ . A standard and useful characterization of stabilization is given in terms of state-space matrices, as follows.

**Proposition 3.** *Suppose  $\mathcal{P}_{22} \in \mathcal{R}_p$  and  $\mathcal{K} \in \mathcal{R}_p$  have the realizations  $(A, B_2, C_2, D_{22})$  and  $(A_K, B_K, C_K, D_K)$  respectively. Then the following statements are equivalent.*

- (i)  $(C_2, A, B_2)$  and  $(C_K, A_K, B_K)$  are stabilizable and detectable, and  $\mathcal{K}$  stabilizes  $\mathcal{P}_{22}$ .
- (ii)  $I - D_{22}D_K$  is invertible and  $\bar{A}$  is Hurwitz, where

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix}$$

**Proof.** That (i) implies (ii) follows from the standard result [30, Lemma 5.2], which states that if  $(C_2, A, B_2)$  and  $(C_K, A_K, B_K)$  are stabilizable and detectable, then  $\mathcal{K}$  stabilizes  $\mathcal{P}_{22}$  if and only if  $\bar{A}$  is Hurwitz. To see the converse, notice that if (ii) holds, then

$$\left( \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix}, \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \right) \text{ is stabilizable}$$

It follows from the PBH test that  $(A, B_2)$  and  $(A_K, B_K)$  are stabilizable. A similar argument proves that  $(C_2, A)$  and  $(C_K, A_K)$  are detectable. Now applying [30, Lemma 5.2] again gives (i) as desired.  $\blacksquare$

Proposition 3 is a characterization of stabilization for the  $\mathcal{P}_{22}$  block. As stated in Proposition 2, we also require the existence of a stabilizing controller for  $\mathcal{P}$ . The following result is standard [30, Lemma 12.1].

**Proposition 4.** *Suppose  $\mathcal{P} \in \mathcal{R}_p$  has a minimal realization given by (5). Then there exists a controller  $\mathcal{K}_0 \in \mathcal{R}_p$  such that  $\mathcal{K}_0$  stabilizes  $\mathcal{P}$  if and only if  $(C_2, A, B_2)$  is stabilizable and detectable. In this case, one such controller is*

$$\mathcal{K}_0 = \left[ \begin{array}{c|c} \frac{A + B_2K + LC_2 + LD_{22}K}{K} & -L \\ \hline & 0 \end{array} \right]$$

where  $K$  and  $L$  are chosen such that  $A + B_2K$  and  $A + LC_2$  are Hurwitz.

We are now ready to state the analogous result in the triangular case.

**Theorem 5.** *Suppose  $\mathcal{P} \in \mathcal{R}_p$  and  $\mathcal{P}_{22} \in \text{lower}(\mathcal{R}_p, k, m)$ . Let  $(A, B, C, D)$  be a minimal realization of  $\mathcal{P}$  that satisfies (5) and (6). There exists  $\mathcal{K}_0 \in \text{lower}(\mathcal{R}_p, m, k)$  such that  $\mathcal{K}_0$  stabilizes  $\mathcal{P}$  if and only if both*

- (i)  $(C_{11}, A_{11}, B_{11})$  is stabilizable and detectable, and
- (ii)  $(C_{22}, A_{22}, B_{22})$  is stabilizable and detectable.



In this case, one such controller is

$$\mathcal{K}_0 = \left[ \begin{array}{c|c} \frac{A + B_2 K_d + L_d C_2 + L_d D_{22} K_d}{K_d} & -L_d \\ \hline & 0 \end{array} \right] \quad (7)$$

where

$$K_d \triangleq \begin{bmatrix} K_1 & \\ & K_2 \end{bmatrix} \quad \text{and} \quad L_d \triangleq \begin{bmatrix} L_1 & \\ & L_2 \end{bmatrix}$$

and  $K_i$  and  $L_i$  are chosen such that  $A_{ii} + B_{ii} K_i$  and  $A_{ii} + L_i C_{ii}$  are Hurwitz for  $i = 1, 2$ .

**Proof.** ( $\Leftarrow$ ) Suppose that (i) and (ii) hold. Note that  $A + B_2 K_d$  and  $A + L_d C_2$  are Hurwitz by construction, thus  $(C_2, A, B_2)$  is stabilizable and detectable. By Proposition 4, the system  $\mathcal{P}$  can be stabilized, and a stabilizing controller is given by (7). Due to the block-diagonal structure of  $K_d$  and  $L_d$ , it is straightforward to verify that  $\mathcal{K}_0 \in \text{lower}(\mathcal{R}_p)$ . ( $\Rightarrow$ ) Suppose a controller  $\mathcal{K}_0 \in \text{lower}(\mathcal{R}_p, m, k)$  stabilizes  $\mathcal{P}$ . By Proposition 4, the realization  $(C_2, A, B_2)$  is stabilizable and detectable. By Theorem 1, we may assume that  $\mathcal{K}_0$  has a minimal realization with  $A_K, B_K, C_K, D_K \in \text{lower}(\mathbb{R})$ . Proposition 3 now implies that  $I - D_{22} D_K$  is invertible. Since  $D_{22}$  and  $D_K$  are block-lower-triangular, write

$$D_{22} \triangleq \begin{bmatrix} \bar{D}_{11} & 0 \\ \bar{D}_{21} & \bar{D}_{22} \end{bmatrix} \quad \text{and} \quad D_K \triangleq \begin{bmatrix} D_{K11} & 0 \\ D_{K21} & D_{K22} \end{bmatrix}$$

Hence we can conclude that  $I - \bar{D}_{11} D_{K11}$  and  $I - \bar{D}_{22} D_{K22}$  are both invertible. Secondly, Proposition 3 also implies that  $\bar{A}$  is Hurwitz. Transform  $\bar{A}$  using a matrix  $T$  that permutes states 2 and 3, and the resulting matrix is block-lower-triangular. Now  $T^{-1} \bar{A} T$  is Hurwitz implies that the two  $2 \times 2$  blocks on the diagonal are Hurwitz. But these diagonal blocks are precisely the  $\bar{A}$  matrices corresponding to the 11 and 22 subsystems. It follows from Proposition 3 that  $(C_{ii}, A_{ii}, B_{ii})$  is stabilizable and detectable for  $i = 1, 2$ . ■

Note that in Theorem 5, the conditions for stabilization are stronger in the decentralized case than those of Proposition 4 for the centralized case. Of course, this is because we have imposed the additional sparsity constraint on the controller. It is also worth observing that one can use the same argument to show that, if there exists a block lower-triangular stabilizing controller, then there also exists a block diagonal stabilizing controller. We also characterize the stability of just the 22 block, which we state as a corollary.

**Corollary 6.** Suppose  $\mathcal{P}_{22} \in \text{lower}(\mathcal{R}_p, k, m)$ , and let  $(A, B_2, C_2, D_{22})$  be a minimal realization of  $\mathcal{P}_{22}$  that satisfies (6). There exists  $\mathcal{K}_0 \in \text{lower}(\mathcal{R}_p, m, k)$  such that  $\mathcal{K}_0$  stabilizes  $\mathcal{P}_{22}$  if and only if both

(i)  $(C_{11}, A_{11}, B_{11})$  is stabilizable and detectable, and

(ii)  $(C_{22}, A_{22}, B_{22})$  is stabilizable and detectable.

**Proof.** ( $\Leftarrow$ ) Augment  $\mathcal{P}_{22}$  by setting  $\mathcal{P}_{11} = 0$ ,  $\mathcal{P}_{12} = 0$ , and  $\mathcal{P}_{21} = 0$ . Then a minimal realization for  $\mathcal{P}$  is given by

$$\mathcal{P} = \left[ \begin{array}{c|cc} A & 0 & B_2 \\ \hline 0 & 0 & 0 \\ C_2 & 0 & D_{22} \end{array} \right]$$

Suppose that (i) and (ii) hold. By Theorem 5, there exists  $\mathcal{K}_0 \in \text{lower}(\mathcal{R}_p)$  which stabilizes  $\mathcal{P}$ , and by Proposition 2,  $\mathcal{K}_0$  also stabilizes  $\mathcal{P}_{22}$ .

( $\Rightarrow$ ) Suppose a controller  $\mathcal{K}_0 \in \text{lower}(\mathcal{R}_p, m, k)$  stabilizes  $\mathcal{P}_{22}$ . Since  $(A, B_2, C_2, D_{22})$  is minimal, we have that  $(C_2, A, B_2)$  is stabilizable and detectable. The result then follows as in the proof of Theorem 5. ■

In the centralized case, every  $\mathcal{P}_{22} \in \mathcal{R}_p$  can be stabilized. Corollary 6 shows that structured stabilization is nontrivial. Indeed, there exist block-lower-triangular transfer matrices that cannot be stabilized by a block-lower-triangular controller. For example, consider

$$\mathcal{P}_{22} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s-1} & \frac{1}{s+1} \end{bmatrix} = \left[ \begin{array}{ccc|cc} -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array} \right]$$

The above realization is minimal, but the grouping of states into blocks is not unique. We can choose to group the unstable mode either in the  $A_{11}$  block or in the  $A_{22}$  block, which corresponds to  $n = (2, 1)$  or  $n = (1, 2)$ , respectively. If we choose the former, it will lead to an undetectable  $(C_{11}, A_{11})$ , and if we choose the latter, it will lead to an unstabilizable  $(A_{22}, B_{22})$ . So by Corollary 6, this plant cannot be stabilized by a block-lower-triangular controller.

One may be tempted to think that a plant containing unstable poles in its off-diagonal block can never be stabilized using a block-lower-triangular controller, but this is false in general. An example is given by the plant and controller pair below.

$$\mathcal{P}_{22} = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{s-1} & \frac{1}{s+1} \end{bmatrix} \quad \mathcal{K}_0 = \begin{bmatrix} \frac{-4}{s+3} & 0 \\ 0 & 0 \end{bmatrix}$$

We now turn our attention to the parameterization of all stabilizing controllers.

**Theorem 7.** *Suppose the conditions of Theorem 5 hold, and  $(C_{11}, A_{11}, B_{11})$  and  $(C_{22}, A_{22}, B_{22})$  are stabilizable and detectable. Define  $K_d$  and  $L_d$  as in Theorem 5. The set of all  $\mathcal{K} \in \text{lower}(\mathcal{R}_p, m, k)$  that stabilize  $\mathcal{P}$  is*

$$\{\mathcal{F}_\ell(\mathcal{J}_d, \mathcal{Q}) \mid \mathcal{Q} \in \text{lower}(\mathcal{RH}_\infty, m, k) \text{ and } I + D_{22}\mathcal{Q}(\infty) \text{ is nonsingular}\}$$

where  $A_d = A + B_2K_d + L_dC_2 + L_dD_{22}K_d$  and

$$\mathcal{J}_d = \left[ \begin{array}{c|cc} A_d & -L_d & B_2 + L_dD_{22} \\ \hline K_d & 0 & I \\ -(C_2 + D_{22}K_d) & I & -D_{22} \end{array} \right] \quad (8)$$

**Proof.** If we relax the constraint that  $\mathcal{Q}$  be lower triangular, then this is the standard parameterization of all centralized stabilizing controllers [30, Theorem 12.8]. It suffices to show that the map from  $\mathcal{Q}$  to  $\mathcal{K}$  and its inverse are structure-preserving. Since each block of the state-space realization of  $\mathcal{J}_d$  is in  $\text{lower}(\mathbb{R})$ , we have  $(\mathcal{J}_d)_{ij} \in \text{lower}(\mathcal{R}_p)$ . Thus  $F_\ell(\mathcal{J}_d, \cdot)$  preserves lower triangularity on its domain. This also holds for the inverse map  $F_u(\mathcal{J}_d^{-1}, \cdot)$ , since

$$\mathcal{J}_d^{-1} = \left[ \begin{array}{c|cc} A & B_2 & -L_d \\ \hline C_2 & D_{22} & I \\ -K_d & I & 0 \end{array} \right] \quad \blacksquare$$

As in the centralized case, this parameterization of stabilizing controllers allows us to transform the controller synthesis problem into a stable model-matching problem. For the two-player problem, the structural constraint we have on the controller will amount to the same structural constraint on the Youla parameter. We have the following algebraic result.

**Theorem 8.** Suppose the conditions of Theorem 7 hold,  $\mathcal{Q} \in \text{lower}(\mathcal{RH}_\infty, m, k)$  and the matrix  $I + D_{22}\mathcal{Q}(\infty)$  is nonsingular. Then if  $\mathcal{K} = F_\ell(\mathcal{J}_d, \mathcal{Q})$  we have

$$F_\ell(\mathcal{P}, \mathcal{K}) = \mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}\mathcal{T}_{21}$$

where  $\mathcal{J}_d$  is given by (8), and  $\mathcal{T}$  is

$$\begin{bmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{bmatrix} = \left[ \begin{array}{cc|cc} A + B_2K_d & -B_2K_d & B_1 & B_2 \\ 0 & A + L_dC_2 & B_1 + L_dD_{21} & 0 \\ \hline C_1 + D_{12}K_d & -D_{12}K_d & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array} \right] \quad (9)$$

**Proof.** This follows from simplifying  $F_\ell(\mathcal{P}, F_\ell(\mathcal{J}_d, \mathcal{Q}))$ . ■

Combining the previous two results gives the following important corollary.

**Corollary 9.** Suppose the conditions of Theorem 7 hold. Suppose  $\mathcal{Q}_{\text{opt}}$  is a minimizer for

$$\begin{aligned} &\text{minimize} && \|\mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}\mathcal{T}_{21}\|_2 \\ &\text{subject to} && \mathcal{Q} \in \text{lower}(\mathcal{RH}_\infty, m, k) \\ &&& I + D_{22}\mathcal{Q}(\infty) \text{ is invertible} \end{aligned} \quad (10)$$

Then choosing  $\mathcal{K}_{\text{opt}} = F_\ell(\mathcal{J}_d, \mathcal{Q}_{\text{opt}})$  gives a minimizer for the two-player output feedback problem (4). Here  $\mathcal{J}_d$  is given by (8), and  $\mathcal{T}$  is defined in (9). Furthermore,  $\mathcal{Q}_{\text{opt}}$  is unique if and only if  $\mathcal{K}_{\text{opt}}$  is unique.

**Remark 10.** In the case where  $\mathcal{P}$  is already stable, a stabilizing controller is found by choosing  $K_d = 0$  and  $L_d = 0$ . The parameterization of Theorem 7 simplifies to

$$\mathcal{K} = F_\ell(\mathcal{J}_d, \mathcal{Q}) = \mathcal{Q}(I + \mathcal{P}_{22}\mathcal{Q})^{-1} \quad (11)$$

Finally, the result of Corollary 9 amounts to solving the model-matching problem where  $\mathcal{T}$  is replaced by  $\mathcal{P}$ , and once we have found  $\mathcal{Q}_{\text{opt}}$ , we can recover  $\mathcal{K}_{\text{opt}}$  via (11).

Using a parameterization akin to that of Youla, Corollary 9 transforms the two-player output feedback problem (4) into a two-player stable model-matching problem (10). The new formulation should be easier to solve than the output feedback version because it is convex and the  $\mathcal{T}_{ij}$  are stable. However, its solution is still not straightforward, because the problem remains infinite-dimensional and there is a structural constraint on  $\mathcal{Q}$ . In the next section, we will show how to solve this problem, and find the solution to (4) in the process.

## IV Main result

In this section, we present our main result: a solution to the two-player output feedback problem. First, we state our assumptions and list the equations that must be solved. Our first assumption is that the plant  $\mathcal{P}$  satisfies

$$\begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} = \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right] \quad \text{is a minimal realization} \quad (12)$$

in which the matrices  $A$ ,  $B_2$ , and  $C_2$  have the form

$$A \triangleq \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad B_2 \triangleq \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \quad C_2 \triangleq \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix} \quad (13)$$

where  $A_{11} \in \mathbb{R}^{n_1 \times n_1}$ ,  $B_{11} \in \mathbb{R}^{n_1 \times m_1}$  and  $C_{11} \in \mathbb{R}^{k_1 \times n_1}$ . If  $\mathcal{P}_{22} \in \text{lower}(\mathcal{R}_p, k, m)$ , then there is no loss in generality in assuming this structure for  $A, B_2, C_2$  because of Theorem 1. We further assume that  $A_{11}$  and  $A_{22}$  have non-empty dimensions. This avoids trivial special cases and allows us to streamline the results. As for the assumptions that  $D_{11} = 0$  and  $D_{22} = 0$ , these assumptions can be made without loss of generality as in the centralized case [30]. To ease notation, we define the following cost and covariance matrices

$$\begin{aligned} \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} &\triangleq \begin{bmatrix} C_1^\top C_1 & C_1^\top D_{12} \\ D_{12}^\top C_1 & D_{12}^\top D_{12} \end{bmatrix} = [C_1 \quad D_{12}]^\top [C_1 \quad D_{12}] \\ \begin{bmatrix} W & U^\top \\ U & V \end{bmatrix} &\triangleq \begin{bmatrix} B_1 B_1^\top & B_1 D_{21}^\top \\ D_{21} B_1^\top & D_{21} D_{21}^\top \end{bmatrix} = \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix}^\top \end{aligned} \quad (14)$$

The main assumptions for the two-player problem are as follows.

- A1)  $D_{12}^\top D_{12} > 0$
- A2)  $(A_{11}, B_{11})$  and  $(A_{22}, B_{22})$  are stabilizable
- A3)  $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$  has full column rank for all  $\omega \in \mathbb{R}$
- A4)  $D_{21} D_{21}^\top > 0$
- A5)  $(C_{11}, A_{11})$  and  $(C_{22}, A_{22})$  are detectable
- A6)  $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$  has full row rank for all  $\omega \in \mathbb{R}$

We will also require the solutions to four AREs

$$\begin{aligned} (X, K) &= \text{ARE}(A, B_2, C_1, D_{12}) & (Y, L^\top) &= \text{ARE}(A^\top, C_2^\top, B_1^\top, D_{21}^\top) \\ (\tilde{X}, J) &= \text{ARE}(A_{22}, B_{22}, C_1 E_2, D_{12} E_2) & (\tilde{Y}, M^\top) &= \text{ARE}(A_{11}^\top, C_{11}^\top, E_1^\top B_1^\top, E_1^\top D_{21}^\top) \end{aligned} \quad (15)$$

Finally, we must solve the following simultaneous linear equations for  $\Phi$  and  $\Psi$

$$\begin{aligned} (A_{22} + B_{22}J)^\top \Phi + \Phi(A_{11} + MC_{11}) - (\tilde{X} - X_{22})(\Psi C_{11}^\top + U_{12}^\top) V_{11}^{-1} C_{11} \\ + (\tilde{X} A_{21} + J^\top S_{12}^\top + Q_{21} - X_{21} M C_{11}) = 0 \end{aligned} \quad (16)$$

$$\begin{aligned} (A_{22} + B_{22}J)\Psi + \Psi(A_{11} + MC_{11})^\top - B_{22} R_{22}^{-1} (B_{22}^\top \Phi + S_{12}^\top) (\tilde{Y} - Y_{11}) \\ + (A_{21} \tilde{Y} + U_{12}^\top M^\top + W_{21} - B_{22} J Y_{21}) = 0 \end{aligned} \quad (17)$$

and define the associated gains  $\hat{K}$  and  $\hat{L}$

$$\hat{K} = \begin{bmatrix} 0 & 0 \\ -R_{22}^{-1} (B_{22}^\top \Phi + S_{12}^\top) & J \end{bmatrix} \quad \hat{L} = \begin{bmatrix} M & 0 \\ -(\Psi C_{11}^\top + U_{12}^\top) V_{11}^{-1} & 0 \end{bmatrix} \quad (18)$$

The matrices  $\Phi$  and  $\Psi$  have physical interpretations, as do the gains  $\hat{K}$  and  $\hat{L}$ . These will be explained in Section V. The main result of this paper is Theorem 11, given below.

**Theorem 11.** *Suppose  $\mathcal{P} \in \mathcal{R}_p$  satisfies (12), Assumptions A1–A6, and the structural condition (13). The two-player output feedback problem*

$$\begin{aligned} &\text{minimize} && \|\mathcal{P}_{11} + \mathcal{P}_{12} \mathcal{K} (I - \mathcal{P}_{22} \mathcal{K})^{-1} \mathcal{P}_{21}\|_2 \\ &\text{subject to} && \mathcal{K} \in \text{lower}(\mathcal{R}_p, m, k) \\ &&& \mathcal{K} \text{ stabilizes } \mathcal{P} \end{aligned}$$

as stated in (4) has a unique solution, given by

$$\mathcal{K}_{\text{opt}} = \left[ \begin{array}{cc|c} A + B_2 K + \hat{L} C_2 & 0 & -\hat{L} \\ B_2 K - B_2 \hat{K} & A + L C_2 + B_2 \hat{K} & -L \\ \hline K - \hat{K} & \hat{K} & 0 \end{array} \right] \quad (19)$$

where  $K$ ,  $L$ ,  $\hat{K}$  and  $\hat{L}$  are defined in (15)–(18).

Note that the sparsity structure of the gains  $\hat{L}$  and  $\hat{K}$  gives  $\mathcal{K}_{\text{opt}}$  the desired block-lower-triangular structure. In the next sections, we will discuss several aspects of this solution. First, a brief discussion on duality and symmetry. We then show a structural result, that the states of the optimal controller have a natural interpretation as minimum-mean-square-error estimates. This will lead to a useful interpretation of the matrices  $\Phi$  and  $\Psi$  that were computed in (16) and (17). We then compute the  $\mathcal{H}_2$ -norm of the optimally controlled system, and characterize the portion of the cost attributed to the decentralization constraint. Finally, we show how our main result specializes to many previously solved cases appearing in the literature.

We provide a complete proof to Theorem 11 below. First, motivated by Corollary 9, we will consider model-matching problems with triangular constraints. We show that under our assumptions, an optimal rational controller exists and is unique. We construct the optimality conditions for the two-player problem and show that (19) satisfies them.

We consider model-matching problems that have stable parameters. Stability is important because it allows us to work in the Hilbert space  $\mathcal{H}_2$  and derive useful optimality conditions. For the rest of this section, we assume the  $\mathcal{T}_{ij} \in \mathcal{RH}_\infty$  satisfy

$$B1) \mathcal{T}_{11}(\infty) = 0$$

$$B2) \mathcal{T}_{12}(j\omega) \text{ has full column rank for all } \omega \in \mathbb{R} \cup \{\infty\}$$

$$B3) \mathcal{T}_{21}(j\omega) \text{ has full row rank for all } \omega \in \mathbb{R} \cup \{\infty\}$$

Model-matching problems with stable parameters have associated optimality conditions that follow from the Hilbert space projection theorem. The optimality condition for the centralized version of the problem is given in the following lemma.

**Lemma 12.** *Suppose  $\mathcal{T} \in \mathcal{RH}_\infty$  satisfies Assumptions B1–B3. Then*

$$\begin{aligned} & \text{minimize} && \|\mathcal{T}_{11} + \mathcal{T}_{12} \mathcal{Q} \mathcal{T}_{21}\|_2 \\ & \text{subject to} && \mathcal{Q} \in \mathcal{RH}_2 \end{aligned} \quad (20)$$

has a unique solution. Furthermore,  $\mathcal{Q}$  is the minimizer of (20) if and only if

$$\mathcal{T}_{12}^* (\mathcal{T}_{11} + \mathcal{T}_{12} \mathcal{Q} \mathcal{T}_{21}) \mathcal{T}_{21}^* \in \mathcal{H}_2^\perp \quad (21)$$

**Proof.** Consider (20) with the relaxed constraint that  $\mathcal{Q} \in \mathcal{H}_2$  instead of  $\mathcal{Q} \in \mathcal{RH}_2$ . The map  $\mathcal{Q} \mapsto \mathcal{T}_{12} \mathcal{Q} \mathcal{T}_{21}$  is a bounded linear operator from  $\mathcal{H}_2$  to  $\mathcal{H}_2$ , so the Hilbert space projection theorem (see, for example [13]) implies that (21) is necessary and sufficient for optimality. If  $\mathcal{Q} \in \mathcal{RH}_2$  is a solution to (20), then the density of rationals imply that it must also be a solution to the relaxed problem and thus it satisfies (21). Conversely, solutions to (21) that happen to be rational must also be solutions to (20). Existence of an optimal rational controller follows from Proposition 24, while uniqueness follows from Assumptions B1–B3. ■

Note that in Corollary 9, the constraint was  $\mathcal{Q} \in \mathcal{RH}_\infty$  rather than  $\mathcal{Q} \in \mathcal{RH}_2$ . There is no loss of generality in assuming  $\mathcal{Q} \in \mathcal{RH}_2$  because of Assumptions B1–B3. Also  $\mathcal{Q}(\infty) = 0$  is necessary to ensure we have a finite  $\mathcal{H}_2$ -norm. In the following lemma, we develop an optimality condition similar to (21) for the two-player model-matching problem.

**Lemma 13.** Suppose  $\mathcal{T} \in \mathcal{RH}_\infty$  satisfies Assumptions B1–B3. Then the two-player model-matching problem

$$\begin{aligned} & \text{minimize} && \|\mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}\mathcal{T}_{21}\|_2 \\ & \text{subject to} && \mathcal{Q} \in \text{lower}(\mathcal{RH}_2) \end{aligned} \quad (22)$$

has a unique solution. Furthermore,  $\mathcal{Q}$  is the minimizer of (22) if and only if

$$\mathcal{T}_{12}^* (\mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}\mathcal{T}_{21}) \mathcal{T}_{21}^* \in \begin{bmatrix} \mathcal{H}_2^\perp & \mathcal{L}_2 \\ \mathcal{H}_2 & \mathcal{H}_2^\perp \end{bmatrix} \quad (23)$$

**Proof.** Since the  $\mathcal{H}_2$ -norm is invariant under rearrangement of matrix elements, we may vectorize [6] the contents of the norm in (22) to obtain

$$\begin{aligned} & \text{minimize} && \|\text{vec}(\mathcal{T}_{11}) + (\mathcal{T}_{21}^\top \otimes \mathcal{T}_{12}) \text{vec}(\mathcal{Q})\|_2 \\ & \text{subject to} && \mathcal{Q} \in \text{lower}(\mathcal{RH}_2) \end{aligned} \quad (24)$$

Due to the sparsity pattern of  $\mathcal{Q}$ , some entries of  $\text{vec}(\mathcal{Q})$  will be zero. Let  $E$  be the identity matrix with columns removed corresponding to these zero-entries. Then  $\mathcal{Q} \in \text{lower}(\mathcal{RH}_2)$  if and only if  $\text{vec}(\mathcal{Q}) = Eq$  for some  $q \in \mathcal{RH}_2$ . Then (24) is equivalent to

$$\begin{aligned} & \text{minimize} && \|\text{vec}(\mathcal{T}_{11}) + (\mathcal{T}_{21}^\top \otimes \mathcal{T}_{12})Eq\|_2 \\ & \text{subject to} && q \in \mathcal{RH}_2 \end{aligned} \quad (25)$$

This is a centralized model-matching problem of the form (20). Using standard properties of the Kronecker product, one may verify that Assumptions B1–B3 are inherited by

$$\begin{bmatrix} \text{vec}(\mathcal{T}_{11}) & (\mathcal{T}_{21}^\top \otimes \mathcal{T}_{12})E \\ 1 & 0 \end{bmatrix}$$

The optimality condition (23), along with existence and uniqueness of the solution, follows from Lemma 12.  $\blacksquare$

The vectorization approach of Lemma 13 not only proves the existence of an optimal controller, it provides a reduction to a centralized problem with a well-known solution; see Proposition 24. Unfortunately, constructing the solution in this manner is not feasible in practice because it requires finding a state-space realization of the Kronecker product system. This leads to a dramatic increase in state dimension, and requires solving a large Riccati equation. Furthermore, we lose any physical interpretation of the states, as mentioned in Section I.

**Proof of Theorem 11.** To verify that  $\mathcal{K}_{\text{opt}}$  is optimal, use the parameterization of Theorem 7 to find the parameter  $\mathcal{Q}_s$  that generates  $\mathcal{K}_{\text{opt}}$ . The computation yields

$$\mathcal{Q}_s = F_u(\mathcal{J}_d^{-1}, \mathcal{K}_{\text{opt}}) = \left[ \begin{array}{cc|c} A + B_2K & 0 & \hat{L} \\ 0 & A + LC_2 & L_d - L \\ \hline K_d - K & \hat{K} & 0 \end{array} \right]$$

We notice that  $\mathcal{Q}_s \in \text{lower}(\mathcal{RH}_2)$ , so by Theorem 7,  $\mathcal{K}_{\text{opt}}$  is an admissible stabilizing controller. Now apply Lemma 13 to the  $\mathcal{T}$  found by the stabilizing transformation (9). After some algebraic manipulation and simplifications, we can verify that

$$\mathcal{T}_{12}^* (\mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}_s\mathcal{T}_{21}) \mathcal{T}_{21}^* \in \begin{bmatrix} \mathcal{H}_2^\perp & \mathcal{L}_2 \\ \mathcal{H}_2 & \mathcal{H}_2^\perp \end{bmatrix}$$

so  $\mathcal{Q}_s$  is the solution to the induced two-player model-matching problem. (A similar algebraic approach is detailed in Section VI.) Furthermore, it is straightforward to check that Assumptions B1–B3 hold in the induced problem as well. It follows by Corollary 9 that  $\mathcal{K}_{\text{opt}}$  is the unique optimal controller for the two-player output feedback problem. ■

The optimal closed-loop map is

$$\mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}_{\text{opt}}(I - \mathcal{P}_{22}\mathcal{K}_{\text{opt}})^{-1}\mathcal{P}_{21} = \mathcal{T}_{11} + \mathcal{T}_{12}\mathcal{Q}_s\mathcal{T}_{21}$$

This result significantly differs from the centralized case in the following way. In the centralized case, choosing the Youla parameter using  $K$  and  $L$  rather than  $K_d$  and  $L_d$  yields a nominal stabilizing controller that also happens to be  $\mathcal{H}_2$ -optimal. So we find that the optimal centralized controller  $\mathcal{K}_{\text{centr}}$  corresponds to  $\mathcal{Q}_{\text{you}} = 0$ . This is not the case for the two-player problem because of the additional structural constraint. Indeed, the nominal stabilizing controller from Theorem 5 has the same state dimension as the plant, whereas the optimal controller has twice the state dimension of the plant.

## V Structure of the optimal controller

### V-A Symmetry and duality

The solution to the two-player output feedback problem presents some nice symmetry properties that are perhaps unexpected given the network topology. Player 2 obtains more information than Player 1, so one might expect Player 2's optimal control policy to be more complicated than that of Player 1. Yet, this is not the case. While Player 2 observes all the measurements, he only controls  $u_2$ , so his influence is limited to his own system. In contrast, Player 1 only observes his own system, but he controls  $u_1$ , which in turn influences both systems. This duality is reflected in (15)–(18). For convenience, we define the Hurwitz matrices

$$\begin{aligned} A_K &\triangleq A + B_2K & A_L &\triangleq A + LC_2 \\ A_J &\triangleq A_{22} + B_{22}J & A_M &\triangleq A_{11} + MC_{11} \\ \hat{A} &\triangleq A + B_2\hat{K} + \hat{L}C_2 \end{aligned} \tag{26}$$

Note that  $A_K, A_L, A_J, A_M$  are all Hurwitz by construction, and  $\hat{A}$  is Hurwitz as well, because it is block-lower-triangular and its block-diagonal entries are  $A_M$  and  $A_J$ .

If we transpose all the system variables and swap Player 1 and Player 2, then every quantity related to control of the original system becomes a corresponding quantity related to estimation of the transformed system. More formally, if we define the operator

$$A^\dagger = \begin{bmatrix} A_{22}^\top & A_{12}^\top \\ A_{21}^\top & A_{11}^\top \end{bmatrix}$$

then the transformation  $(A, B_2, C_2) \mapsto (A^\dagger, C_2^\dagger, B_2^\dagger)$  leads to

$$(X, K) \mapsto (Y^\dagger, L^\dagger), \quad (\hat{X}, \hat{K}) \mapsto (\hat{Y}^\dagger, \hat{L}^\dagger), \quad \text{and} \quad \hat{A} \mapsto \hat{A}^\dagger.$$

Note as well that the realization (19) has a dual representation given by

$$\mathcal{K}_{\text{opt}} = \left[ \begin{array}{cc|c} A + B_2K + \hat{L}C_2 & 0 & \hat{L} \\ LC_2 - \hat{L}C_2 & A + LC_2 + B_2\hat{K} & L - \hat{L} \\ \hline -K & -\hat{K} & 0 \end{array} \right] \tag{27}$$

## V-B Estimation structure

Recall the plant equations, written as differential equations

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ y &= C_2 x + D_{21} w \end{aligned} \quad (28)$$

The optimal controller (19) is given in Theorem 11, and we label its states as  $\zeta$  and  $\xi$ , and place each equation into canonical observer form.

$$\dot{\zeta} = A\zeta + B_2 \hat{u} - \hat{L}(y - C_2 \zeta) \quad (29)$$

$$\hat{u} = K\zeta \quad (30)$$

$$\dot{\xi} = A\xi + B_2 u - L(y - C_2 \xi) \quad (31)$$

$$u = K\zeta + \hat{K}(\xi - \zeta) \quad (32)$$

This choice of coordinates leads to a natural interpretation, that  $\zeta$  and  $\xi$  are the states of steady-state Kalman filters that estimate the plant's state  $x$ . This claim will be made precise in the development that follows.

**Lemma 14.** *We make use of the definitions of (12)–(18) and (26). There exists a unique matrix  $\hat{Y}$  satisfying the Lyapunov equation*

$$\hat{A}(\hat{Y} - Y) + (\hat{Y} - Y)\hat{A}^\top + (\hat{L} - L)V(\hat{L} - L)^\top = 0 \quad (33)$$

Further,  $\hat{Y} \geq Y$ ,  $\hat{Y}_{11} = \tilde{Y}$ ,  $\hat{Y}_{21} = \Psi$ , and

$$\hat{L} = -(\hat{Y}C_2^\top + U^\top)E_1V_{11}^{-1}E_1^\top \quad (34)$$

**Proof.** Since  $\hat{A}$  is stable and  $(\hat{L} - L)V(\hat{L} - L)^\top \geq 0$ , it follows from standard properties of Lyapunov equations that (33) has a unique solution  $\hat{Y} - Y \geq 0$ . Right-multiplying by  $E_1$  gives

$$\hat{A}(\hat{Y}E_1 - YE_1) + (\hat{Y}E_1 - YE_1)\hat{A}_M^\top + (\hat{L} - L)V(E_1M^\top - L^\top E_1) = 0$$

Upon comparison with (15), (17), and (18), one can verify that this equation is satisfied by  $\hat{Y}_{11} = \tilde{Y}$  and  $\hat{Y}_{21} = \Psi$ . Similarly comparing with (18) verifies (34). ■

Note that (34) provides a way of computing  $\hat{L}$  that is equivalent to our original definitions in (18). Our next observation is that under the right coordinates, the controllability Gramian of the closed-loop map is block-diagonal.

**Theorem 15.** *Suppose we have the plant described by (28) and the optimal controller given by (29)–(32). The closed-loop map for one particular choice of coordinates is*

$$\begin{bmatrix} \dot{\zeta} \\ \dot{\xi} - \dot{\zeta} \\ \dot{x} - \dot{\xi} \end{bmatrix} = \begin{bmatrix} A_K & -\hat{L}C_2 & -\hat{L}C_2 \\ 0 & \hat{A} & (\hat{L} - L)C_2 \\ 0 & 0 & A_L \end{bmatrix} \begin{bmatrix} \zeta \\ \xi - \zeta \\ x - \xi \end{bmatrix} + \begin{bmatrix} -\hat{L}D_{21} \\ (\hat{L} - L)D_{21} \\ B_1 + LD_{21} \end{bmatrix} w$$

which we write compactly as  $\dot{q} = A_c q + B_c w$ . Let  $\Theta$  be the controllability Gramian, i.e. the solution  $\Theta > 0$  to the Lyapunov equation  $A_c \Theta + \Theta A_c^\top + B_c B_c^\top = 0$ . Then

$$\Theta = \begin{bmatrix} Z & 0 & 0 \\ 0 & \hat{Y} - Y & 0 \\ 0 & 0 & Y \end{bmatrix}$$

where  $Z$  satisfies the Lyapunov equation

$$A_K Z + Z A_K^\top + \hat{L} V \hat{L}^\top = 0$$



**Proof.** Uniqueness and positivity of  $\Theta$  follows because  $A_c$  is Hurwitz. The solution can be verified by direct substitution and by making use of the identities in Lemma 14. ■

The observation in Theorem 15 lends itself to a statistical interpretation. If  $w$  is IID white Gaussian noise, and we consider the steady-state distribution of the random vector  $q$ , its three components are mutually independent. Since  $\xi$  is the sum of the first two states, we have that  $\xi$  and  $x - \xi$  are independent, and similarly  $\zeta$  and  $x - \zeta$  are also independent. This fact supports our claim that  $\zeta$  and  $\xi$  are estimators for  $x$ , and we will provide a proof in the development that follows.

As expected from our discussion on duality, the estimation structure discussed in Lemma 14 and Theorem 15 above has an analogous control structure. We state the analogue to Lemma 14 without proof.

**Lemma 16.** *We make use of the definitions of (12)–(18) and (26). There exists a unique matrix  $\hat{X}$  satisfying the Lyapunov equation*

$$\hat{A}^\top(\hat{X} - X) + (\hat{X} - X)\hat{A} + (\hat{K} - K)^\top R(\hat{K} - K) = 0 \quad (35)$$

Further,  $\hat{X} \geq X$ ,  $\hat{X}_{22} = \tilde{X}$ ,  $\hat{X}_{21} = \Phi$ , and

$$\hat{K} = -E_2 R_{22}^{-1} E_2^\top (B_2^\top \hat{X} + S^\top) \quad (36)$$

The dual of Theorem 15 can be obtained by expressing the closed-loop map in the coordinates  $(x, x - \zeta, x - \xi)$  instead. In these coordinates, one can show that the observability Gramian is block-diagonal.

Now, we prove some structural properties of the optimal controller. Most importantly, we show that the states  $\zeta$  and  $\xi$  of the optimal controller can be interpreted as Kalman filters. We begin by formally defining a notation that was first used in the introduction.

**Definition 17.** Suppose  $\mathcal{G} \in \mathcal{R}_p^{(p_1+p_2) \times (q_1+q_2)}$  is given by

$$\mathcal{G} = \begin{bmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{bmatrix}$$

and  $\mathcal{G}_{21}(j\omega)$  has full row rank for all  $\omega \in \mathbb{R} \cup \{\infty\}$ . Define the estimator of  $\mathcal{G}$  to be  $\mathcal{G}^{\text{est}} \in \mathcal{R}_p^{p_1 \times (p_2+q_1)}$  partitioned according to  $\mathcal{G}^{\text{est}} = [\mathcal{G}_1^{\text{est}} \quad \mathcal{G}_2^{\text{est}}]$  where

$$\begin{aligned} \mathcal{G}_1^{\text{est}} &= \arg \min_{\substack{\mathcal{F} \in \mathcal{RH}_2 \\ \mathcal{G}_{11} - \mathcal{F}\mathcal{G}_{21} \in \mathcal{RH}_2}} \|\mathcal{G}_{11} - \mathcal{F}\mathcal{G}_{21}\|_{\mathcal{H}_2} \\ \mathcal{G}_2^{\text{est}} &= \mathcal{G}_{12} - \mathcal{G}_1^{\text{est}} \mathcal{G}_{22} \end{aligned}$$

Note that under the assumptions of Definition 17, the estimator  $\mathcal{G}^{\text{est}}$  is unique. We will show existence of  $\mathcal{G}^{\text{est}}$  for particular  $\mathcal{G}$  below. We now define the following notation.

**Definition 18.** Suppose  $\mathcal{G}$  and  $\mathcal{G}^{\text{est}}$  are as in Definition 17. If

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathcal{G} \begin{bmatrix} w \\ u \end{bmatrix}$$

then we use the notation  $x_{|y,u}$  to mean

$$x_{|y,u} = \mathcal{G}^{\text{est}} \begin{bmatrix} y \\ u \end{bmatrix}$$

This notation is motivated by the property that

$$x - x_{|y,u} = (\mathcal{G}_{11} - \mathcal{G}_1^{\text{est}} \mathcal{G}_{21})w$$

and hence we are choosing  $\mathcal{G}_1^{\text{est}}$  to minimize the mean square estimation error. In the following lemma we compute the estimator from Definitions 17 and 18 for the centralized case and show that this is exactly the classical steady-state Kalman filter. Specifically, this result shows that  $x_{|y,u}$  is the usual steady-state Kalman filter for estimating the state  $x$  using the measurements  $y$  and inputs  $u$ .

**Lemma 19.** *Let  $\mathcal{G}$  be*

$$\mathcal{G} = \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline I & 0 & 0 \\ C_2 & D_{21} & 0 \end{array} \right]$$

*and suppose Assumptions A4–A6 hold. Then*

$$\mathcal{G}^{\text{est}} = \left[ \begin{array}{c|cc} A_L & -L & B_2 \\ \hline I & 0 & 0 \end{array} \right]$$

*where  $L$  is given by (15).*

**Proof.** Let  $\mathcal{G}_1^{\text{est}}$  and  $L$  be as above. For the optimization problem of Definition 17, change variables according to  $\bar{\mathcal{F}} = \mathcal{G}_1^{\text{est}} + \bar{\mathcal{F}}$ . Then, in these coordinates, the optimization problem is

$$\begin{aligned} & \text{minimize} && \left\| \left[ \begin{array}{c|cc} A_L & B_1 + LD_{21} \\ \hline I & 0 \end{array} \right] - \bar{\mathcal{F}} \mathcal{G}_{21} \right\|_2 \\ & \text{subject to} && \bar{\mathcal{F}} \in \mathcal{RH}_2 \end{aligned}$$

together with the additional constraint that the quantity in the norm is stable. For this proof, it suffices to show that  $\bar{\mathcal{F}} = 0$  is optimal. We begin by parameterizing all  $\bar{\mathcal{F}} \in \mathcal{RH}_2$  such that the quantity in the norm is stable. By [30, Thm. 13.35], the following factorization is coprime

$$\mathcal{G}_{21} = \left[ \begin{array}{c|c} A_L & L \\ \hline C_2 & I \end{array} \right]^{-1} \left[ \begin{array}{c|cc} A_L & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array} \right] = \mathcal{M}^{-1} \mathcal{N}$$

and  $\mathcal{N}\mathcal{N}^* = D_{21}D_{21}^\top$ . Hence the set of  $\bar{\mathcal{F}} \in \mathcal{RH}_2$  such that

$$\bar{\mathcal{F}} \mathcal{G}_{21} \in \mathcal{RH}_2 \tag{37}$$

is parameterized by  $\bar{\mathcal{F}} = \mathcal{Q}\mathcal{M}$  where  $\mathcal{Q} \in \mathcal{RH}_2$ . To see why, note that if  $\bar{\mathcal{F}} \in \mathcal{RH}_2$  and (37) holds then

$$\bar{\mathcal{F}} \begin{bmatrix} I & \mathcal{M}^{-1}\mathcal{N} \end{bmatrix} \in \mathcal{RH}_2 \implies \bar{\mathcal{F}} \mathcal{M}^{-1} \begin{bmatrix} \mathcal{M} & \mathcal{N} \end{bmatrix} \in \mathcal{RH}_2 \implies \bar{\mathcal{F}} \mathcal{M}^{-1} \in \mathcal{RH}_2$$

So  $\bar{\mathcal{F}} = \mathcal{Q}\mathcal{M}$  for some  $\mathcal{Q} \in \mathcal{RH}_2$ . Conversely, if  $\bar{\mathcal{F}} = \mathcal{Q}\mathcal{M}$ , then

$$\bar{\mathcal{F}} \mathcal{G}_{21} = (\mathcal{Q}\mathcal{M})(\mathcal{M}^{-1}\mathcal{N}) = \mathcal{Q}\mathcal{N} \in \mathcal{RH}_2$$

So finally, our optimization problem takes the following unconstrained form

$$\begin{aligned} & \text{minimize} && \left\| \left[ \begin{array}{c|cc} A_L & B_1 + LD_{21} \\ \hline I & 0 \end{array} \right] - \mathcal{Q} \left[ \begin{array}{c|cc} A_L & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array} \right] \right\|_2 \\ & \text{subject to} && \mathcal{Q} \in \mathcal{RH}_2 \end{aligned}$$

We conclude that  $\mathcal{Q} = 0$  is optimal, and hence  $\bar{\mathcal{F}} = 0$ , because

$$\left[ \begin{array}{c|c} A_L & B_1 + LD_{21} \\ \hline I & 0 \end{array} \right] \left[ \begin{array}{c|c} A_L & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array} \right]^* \in \mathcal{H}_2^\perp \quad (38)$$

Comparing the result of Lemma 19 with the  $\xi$ -state of the optimal two-player controller 31, we notice the following consequence.

**Corollary 20.** *Suppose the conditions of Theorem 11 are satisfied, and label the states of the controller as in (29)–(32). Then*

$$x_{|y,u} = \xi$$

where the estimator is defined by the map  $\mathcal{G} : (w, u) \mapsto (x, y)$  induced by the plant (28).

The above result means that  $\xi$ , one of the states of the optimal controller, is the usual Kalman filter estimate of  $x$  given  $y$  and  $u$ . The next result is more difficult, and gives the analogous result for the other states. The next theorem is our main structural result. We show that  $\zeta$  may also be interpreted as an optimal estimator in the sense of Definition 17 and 18.

**Theorem 21.** *Suppose the conditions of Theorem 11 are satisfied, and label the states of the controller as in (29)–(32). Define also*

$$\begin{aligned} u_\zeta &= (K - \hat{K})\zeta \\ u_\xi &= \hat{K}\xi \end{aligned} \quad (39)$$

so that  $u = u_\zeta + u_\xi$ . Then

$$\begin{bmatrix} x \\ \xi \\ u \end{bmatrix}_{|y_1, u_\zeta} = \begin{bmatrix} \zeta \\ \hat{\zeta} \\ \hat{u} \end{bmatrix}$$

Here the estimator is defined by the map  $\mathcal{G} : (w, u_\zeta) \mapsto (x, \xi, u, y_1)$  induced by (39), the plant (28), and the controller (29)–(32).

**Proof.** The proof parallels that of Lemma 19, so we omit most of the details. Straight-forward algebra gives

$$\mathcal{G} = \left[ \begin{array}{cc|cc} A & B_2\hat{K} & B_1 & B_2 \\ -LC_2 & A + LC_2 + B_2\hat{K} & -LD_{21} & B_2 \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & \hat{K} & 0 & I \\ E_1^\top C_2 & 0 & E_1^\top D_{21} & 0 \end{array} \right]$$

After substituting  $\mathcal{F} = \mathcal{G}_1^{\text{est}} + \bar{\mathcal{F}}$ , computing the coprime factorization, and changing to more convenient coordinates, the resulting unconstrained optimization problem is

$$\begin{aligned} & \text{minimize} \quad \left\| \left[ \begin{array}{cc|c} \hat{A} & (\hat{L} - L)C_2 & (\hat{L} - L)D_{21} \\ 0 & A_L & B_2 + LD_{21} \\ \hline I & I & 0 \\ I & 0 & 0 \\ K & 0 & 0 \end{array} \right] - \mathcal{Q}\mathcal{D} \right\|_2 \\ & \text{subject to} \quad \mathcal{Q} \in \mathcal{RH}_2 \end{aligned}$$

where  $\hat{A}$  is defined in (26), and

$$\mathcal{D} = \left[ \begin{array}{cc|c} \hat{A} & (\hat{L} - L)C_2 & (\hat{L} - L)D_{211} \\ 0 & A_L & B_2 + LD_{21} \\ \hline E_1^\top C_2 & E_1^\top C_2 & E_1^\top D_{21} \end{array} \right]$$

Optimality of  $\mathcal{Q} = 0$  is established by verifying the analogous orthogonality relationship to (38). This is easily done using the Gramian identity from Theorem 15. Then we have

$$\mathcal{G}^{\text{est}} = \left[ \begin{array}{c|cc} \hat{A} & -\hat{L}E_1 & B_2 \\ \hline I & 0 & 0 \\ I & 0 & 0 \\ K & 0 & 0 \end{array} \right]$$

Comparing with (29), we can see that

$$\begin{bmatrix} \zeta \\ \zeta \\ \hat{u} \end{bmatrix} = \mathcal{G}^{\text{est}} \begin{bmatrix} y_1 \\ u_\zeta \end{bmatrix}$$

and the result follows. ■

The result of Theorem 21 that  $\zeta = x_{|y_1, u_\zeta}$  has a clear interpretation that  $\zeta$  is a Kalman filter of  $x$ . However, this assumes that Player 2 is implementing the optimal policy for  $u_2$ . This is important because the optimal estimator gain depends explicitly on this choice of policy. In contrast, in the centralized case, the optimal estimation gain does not depend on the choice of control policy.

Note that using  $u_1$  or  $\hat{u}$  instead of  $u_\zeta$  in Theorem 21 yields an incorrect interpretation of  $\zeta$ . In these coordinates,

$$\begin{aligned} \zeta &= \left[ \begin{array}{cc|cc} A + B_2 E_2 E_2^\top K + \hat{L} C_2 & -\hat{L} E_1 & B_2 E_1 \\ \hline I & 0 & 0 \end{array} \right] \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} \quad \text{or} \\ \zeta &= \left[ \begin{array}{cc|cc} A + B_2 K + \hat{L} C_2 & -\hat{L} E_1 & B_2 \\ \hline I & 0 & 0 \end{array} \right] \begin{bmatrix} y_1 \\ \hat{u} \end{bmatrix} \end{aligned}$$

In general, neither of these maps is stable, so for these choices of signals,  $\zeta$  cannot be an estimator of  $x$  in the sense of Definitions 17 and 18.

We can now state the controller in a particularly simple form. The controller is

$$u = Kx_{|y_1, u_\zeta} + \begin{bmatrix} 0 & 0 \\ H & J \end{bmatrix} (x_{|y, u} - x_{|y_1, u_\zeta}) \quad (40)$$

where  $H = \hat{K}_{21}$ . These equations make clear that Player 1 is using the same control action that it would use if the controller was centralized with both players only measuring  $y_1$ . We can also see that the control action of Player 2 has an additional correction term, given by the gain matrix  $\begin{bmatrix} H & J \end{bmatrix}$  multiplied by the difference between the players' estimates of the states. Note that  $u_\zeta$  is a function only of  $y_1$ , as can be seen from the state-space

$$\begin{aligned} \dot{\zeta} &= A_K \zeta - \hat{L} E_1 (y_1 - [C_{11} \quad 0] \zeta) \\ u_\zeta &= (K - \hat{K}) \zeta \end{aligned}$$

The orthogonality relationships of the form (38) used in Lemma 19 and Theorem 21 to solve the  $\mathcal{H}_2$  optimization problems may also be interpreted in terms of signals in the optimally controlled closed-loop.

**Corollary 22.** Suppose  $u, w, x$  and  $y$  satisfy (28)–(32). Then the maps  $E_2 : w \mapsto x - \xi$  and  $R_2 : w \mapsto y - C_2\xi$ , which give the error and residual for Player 2, are

$$E_2 = \left[ \begin{array}{c|c} A_L & B_1 + LD_{21} \\ \hline I & 0 \end{array} \right] \quad R_2 = \left[ \begin{array}{c|c} A_L & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array} \right]$$

The maps  $E_1 : w \mapsto x - \zeta$  and  $R_1 : w \mapsto y_1 - C_{11}\zeta_1$ , which give the error and residual for Player 1, are

$$E_1 = \left[ \begin{array}{cc|c} \hat{A} & (\hat{L} - L)C_2 & (\hat{L} - L)D_{21} \\ 0 & A_L & B_1 + LD_{21} \\ \hline I & I & 0 \end{array} \right] \quad R_1 = \left[ \begin{array}{c|c} A_M & ME_1^\top - E_1^\top L \\ \hline C_{11} & E_1^\top \end{array} \right] R_2$$

Further, the orthogonality conditions  $E_2 R_2^* \in \mathcal{H}_2^\perp$  and  $E_1 R_1^* \in \mathcal{H}_2^\perp$  are satisfied.

As with Theorem 15, Corollary 22 lends itself to a statistical interpretation. If  $w$  is IID white Gaussian noise, and we consider the steady-state distributions of the error and residual for the second player,  $x - \xi$  and  $y - C_2\xi$  respectively, then they are independent. Similarly, the error and residual for the first player,  $x - \zeta$  and  $y_1 - C_{11}\zeta_1$ , are also independent.

### V-C Optimal cost

We now compute the cost associated with the optimal control policy for the two-player output feedback problem. From centralized  $\mathcal{H}_2$  theory [30], there are many equivalent expressions for the optimal centralized cost. In particular,

$$\begin{aligned} \|F_\ell(\mathcal{P}, \mathcal{K}_{\text{centr}})\|_2^2 &= \left\| \left[ \begin{array}{c|c} A_K & B_1 \\ \hline C_1 + D_{12}K & 0 \end{array} \right] \right\|_2^2 + \left\| \left[ \begin{array}{c|c} A_L & B_1 + LD_{21} \\ \hline D_{12}K & 0 \end{array} \right] \right\|_2^2 \\ &= \text{trace}(XW) + \text{trace}(YK^\top RK) \\ &= \text{trace}(YQ) + \text{trace}(XLVL^\top) \end{aligned}$$

where  $X, Y, K, L$  are defined in (15). Of course, the cost of the optimal two-player controller will be greater, so we may write

$$\|F_\ell(\mathcal{P}, \mathcal{K}_{\text{opt}})\|_2^2 = \|F_\ell(\mathcal{P}, \mathcal{K}_{\text{centr}})\|_2^2 + \Delta$$

where  $\Delta \geq 0$  is the additional cost incurred by decentralization. We now give some closed-form expressions for  $\Delta$  that are similar to the centralized formulae above.

**Theorem 23.** The additional cost incurred by the optimal controller (19) for the two-player problem (4) as compared to the cost of the optimal centralized controller is

$$\begin{aligned} \Delta &= \left\| \left[ \begin{array}{c|c} \hat{A} & (\hat{L} - L)D_{21} \\ \hline D_{12}(\hat{K} - K) & 0 \end{array} \right] \right\|_2^2 \\ &= \text{trace}(\hat{Y} - Y)(\hat{K} - K)^\top R(\hat{K} - K) \\ &= \text{trace}(\hat{X} - X)(\hat{L} - L)V(\hat{L} - L)^\top \end{aligned}$$

where  $K, \hat{K}, L, \hat{L}$  are defined in (15)–(18),  $\hat{X}$  and  $\hat{Y}$  are defined in (33) and (35), and  $\hat{A}$  is defined in (26).

**Proof.** The key is to view  $\mathcal{K}_{\text{opt}}$  as a sub-optimal centralized controller. Centralized  $\mathcal{H}_2$  theory [30] then implies that

$$\Delta = \|D_{12}\mathcal{Q}_{\text{you}}D_{21}\|_2^2 \quad (41)$$

where  $\mathcal{Q}_{\text{you}}$  is the centralized Youla parameter. Specifically,  $\mathcal{Q}_{\text{you}} = F_u(\mathcal{J}^{-1}, \mathcal{K}_{\text{opt}})$  and

$$\mathcal{J}^{-1} = \left[ \begin{array}{c|cc} A & B_2 & -L \\ \hline C_2 & 0 & I \\ -K & I & 0 \end{array} \right]$$

Since this is the centralized parameterization, it contains the optimal centralized gains  $K$  and  $L$  instead of  $K_d$  and  $L_d$ . After some simplification, we obtain

$$\mathcal{Q}_{\text{you}} = \left[ \begin{array}{c|c} \hat{A} & \hat{L} - L \\ \hline \hat{K} - K & 0 \end{array} \right] \quad (42)$$

substituting (42) into (41) yields the first formula. The second formula follows from evaluating (41) in a different way. Note that  $\|D_s + C_s(sI - A_s)^{-1}B_s\|^2 = \text{trace}(C_s W_c C_s^\top)$ , where  $W_c$  is the associated controllability Gramian, given by  $A_s W_c + W_c A_s^\top + B_s B_s^\top = 0$ . In the case of (41), the controllability Gramian equation is precisely (33), and therefore  $W_c = \hat{Y} - Y$  and the second formula follows. The third formula follows using the observability Gramian  $W_o = \hat{X} - X$  given by (35). ■

Theorem 23 expresses the total cost as a sum of two parts; the cost of the optimal centralized controller, and the additional cost incurred by requiring the controller to be block-lower-triangular. This allows us to precisely quantify the cost of decentralization.

## V-D Some special cases

The optimal controller (19) is given by (40), which we repeat here for convenience.

$$u = Kx_{|y_1, u_\zeta} + \begin{bmatrix} 0 & 0 \\ H & J \end{bmatrix} (x_{|y, u} - x_{|y_1, u_\zeta})$$

Recall that  $K$  and  $J$  are found by solving standard AREs (15). The coupling between estimation and control appears in the term  $H = -R_{22}^{-1}(B_{22}^\top \Phi + S_{12}^\top)$ , which is found by solving the coupled linear equations (16)–(17).

Several special cases of the two-player output feedback problem have previously been solved. In each of them, the first component of  $(x_{|y, u} - x_{|y_1, u_\zeta})$  is zero. In other words, both players maintain identical estimates of  $x_1$  given their respective information. Consequently,  $u$  no longer depends on  $H$  and there is no need to compute  $\Phi$  and  $\Psi$ . We now examine these special cases in more detail.

*Centralized.* The problem becomes centralized when both players have access to the same information. In this case, both players maintain identical estimates of the entire state  $x$ . Thus,  $x_{|y, u} = x_{|y_1, u_\zeta} = \hat{x}$  and we recover the well-known centralized solution  $u = K\hat{x}$ .

*State-feedback.* The state-feedback problem for two players is the case where our measurements are noise-free, so that  $y_1 = x_1$  and  $y_2 = x_2$ . Therefore, both players know  $x_1$  exactly. This case is solved in [21, 22] and the solution takes the following form, which agrees with our general formula.

$$u = K \begin{bmatrix} x_1 \\ \hat{x}_{2|1} \end{bmatrix} + \begin{bmatrix} 0 \\ J \end{bmatrix} (x_2 - \hat{x}_{2|1})$$

Here,  $\hat{x}_{2|1}$  is an estimate of  $x_2$  given the information available to Player 1, as stated in [22].

*Partial output feedback.* In the partial-output feedback case,  $y_1 = x_1$  as in the state-feedback case, but  $y_2$  is a noisy linear measurement of both states. In this case, both players know  $x_1$  exactly. This case is solved in [23] and the solution takes the following form (using notation from [23]), which agrees with our general formula.

$$u = K \begin{bmatrix} x_1 \\ \hat{x}_{2|1} \end{bmatrix} + \begin{bmatrix} 0 \\ J \end{bmatrix} (\hat{x}_{2|2} - \hat{x}_{2|1})$$

*Dynamically decoupled.* In the dynamically decoupled case, all measurements are noisy, and the dynamics of both systems are decoupled. This amounts to the case where  $A_{21} = 0$ ,  $B_{21} = 0$ ,  $C_{21} = 0$ , and  $W$ ,  $V$ ,  $U$  are block-diagonal. Due to the decoupled dynamics, the estimate of  $x_1$  based on  $y_1$  does not improve when additionally using  $y_2$ . This case is solved in [7] and the solution takes the following form, which agrees with our general formula.

$$u = K \begin{bmatrix} \hat{x}_{1|1} \\ \hat{x}_{2|1} \end{bmatrix} + \begin{bmatrix} 0 \\ J \end{bmatrix} (\hat{x}_{2|2} - \hat{x}_{2|1})$$

Note that estimation and control are decoupled in all the special cases examined above. This fact allows the optimal controller to be computed by merely solving some subset of the AREs (15). In the general case however, estimation and control are coupled via  $\Phi$  and  $\Psi$  in (16)–(17).

## VI Construction of the optimal controller

The proof of Theorem 11 verifies that the given formula for  $\mathcal{K}_{\text{opt}}$  is actually optimal. In this section, we give a state-space method for constructing it, via the solution to the two-player model-matching problem (22). First, we give state-space versions of Assumptions B1–B3. For matrices  $A, B, C, D$ , define the following conditions.

$$C1) \ D_{12}^T D_{12} > 0$$

$$C2) \ \begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} \text{ has full column rank for all } \omega \in \mathbb{R}$$

$$C3) \ D_{21} D_{21}^T > 0$$

$$C4) \ \begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} \text{ has full row rank for all } \omega \in \mathbb{R}$$

Suppose the transfer matrix  $\mathcal{T} \in \mathcal{RH}_\infty$  has realization

$$\begin{bmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & 0 \end{bmatrix} = \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right] \quad \text{where } A \text{ is Hurwitz.} \quad (43)$$

Then  $\mathcal{T}$  satisfies Assumptions B1–B3 if and only if  $A, B, C, D$  satisfy Assumptions C1–C4, so these assumptions depend only on the transfer function, not the particular realization. We will borrow a great deal of notation from Section IV. We will define  $Q, R, S, W, V, U$  as in (14), and  $X, Y, K, L, \tilde{X}, \tilde{Y}, J, M$  as in (15). Note that these quantities are now defined in terms of the  $A, B, C, D$  from (43).

The centralized model-matching problem (20) has a well-known state-space solution, which we state as a Proposition.

**Proposition 24.** Suppose  $\mathcal{T} \in \mathcal{RH}_\infty$  satisfies (43) and Assumptions C1–C4. The solution to the centralized model-matching problem (20) is

$$\mathcal{Q}_{\text{opt}} = \left[ \begin{array}{cc|c} A_K & B_2 K & 0 \\ 0 & A_L & -L \\ \hline K & K & 0 \end{array} \right]$$

where  $K, L$  are defined in (15), and  $A_K, A_L$  are defined in (26).

We will derive the solution to the two-player model-matching problem by splitting it into two coupled optimization problems. We show that the problem is equivalent to a pair of coupled centralized model-matching problems which must simultaneously be solved. Once we have found this solution, we apply Corollary 9 to obtain the solution to the two-player output feedback problem.

**Lemma 25.** Suppose  $\mathcal{T} \in \mathcal{RH}_\infty$  satisfies (43) and Assumptions C1–C4. Further suppose that  $\mathcal{Q} \in \text{lower}(\mathcal{RH}_2)$  and

$$\mathcal{Q} = \begin{bmatrix} \mathcal{Q}_{11} & 0 \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix}$$

Then  $\mathcal{Q}$  is optimal for the two-player model-matching problem (22) if and only if both of the following hold

$$\|\mathcal{T}_{11} + \mathcal{T}_{12} \mathcal{Q} \mathcal{T}_{21}\|_2 = \min_{\mathcal{Z}_{21}, \mathcal{Z}_{22} \in \mathcal{RH}_2} \left\| \mathcal{T}_{11} + \mathcal{T}_{12} \begin{bmatrix} \mathcal{Q}_{11} & 0 \\ \mathcal{Z}_{21} & \mathcal{Z}_{22} \end{bmatrix} \mathcal{T}_{21} \right\|_2 \quad (44)$$

$$\|\mathcal{T}_{11} + \mathcal{T}_{12} \mathcal{Q} \mathcal{T}_{21}\|_2 = \min_{\mathcal{Z}_{11}, \mathcal{Z}_{21} \in \mathcal{RH}_2} \left\| \mathcal{T}_{11} + \mathcal{T}_{12} \begin{bmatrix} \mathcal{Z}_{11} & 0 \\ \mathcal{Z}_{21} & \mathcal{Q}_{22} \end{bmatrix} \mathcal{T}_{21} \right\|_2 \quad (45)$$

**Proof.** We will prove this lemma by showing that the optimality condition for (22), which is given by Lemma 13, is satisfied if and only if the optimality conditions for (44) and (45) hold. Observe that

$$\|\mathcal{T}_{11} + \mathcal{T}_{12} \mathcal{Q} \mathcal{T}_{21}\|_2 = \|(\mathcal{T}_{11} + \mathcal{T}_{12} E_1 \mathcal{Q}_{11} E_1^\top \mathcal{T}_{21}) + \mathcal{T}_{12} E_2 [\mathcal{Q}_{21} \quad \mathcal{Q}_{22}] \mathcal{T}_{21}\|_2 \quad (46)$$

The minimization problem in (44) has no structural constraint, and so (46) and Lemma 12 imply that (44) is equivalent to

$$E_2^\top \mathcal{T}_{12}^* (\mathcal{T}_{11} + \mathcal{T}_{12} \mathcal{Q} \mathcal{T}_{21}) \mathcal{T}_{21}^* \in [\mathcal{H}_2^\perp \quad \mathcal{H}_2^\perp] \quad (47)$$

Similarly (45) is equivalent to

$$\mathcal{T}_{12}^* (\mathcal{T}_{11} + \mathcal{T}_{12} \mathcal{Q} \mathcal{T}_{21}) \mathcal{T}_{21}^* E_1 \in \begin{bmatrix} \mathcal{H}_2^\perp \\ \mathcal{H}_2^\perp \end{bmatrix} \quad (48)$$

It is immediate that  $\mathcal{Q}$  satisfies both (47) and (48) if and only if it satisfies (23).  $\blacksquare$

Lemma 25 shows that solving the two-player model-matching problem (22) amounts to solving two centralized model-matching problems that are coupled. The first one (44) is an optimization in which we are holding  $\mathcal{Q}_{11}$  fixed and optimizing over the remaining  $\mathcal{Q}_{21}$  and  $\mathcal{Q}_{22}$ . In the following lemma, we solve this optimization problem and give a state-space expression for the optimal  $[\mathcal{Q}_{21} \quad \mathcal{Q}_{22}]$  as a function of  $\mathcal{Q}_{11}$ .

To simplify the required algebra, we will assume that in the realization for  $\mathcal{T}$  given by (43), the matrices  $A, B_2, C_2$  have the block-lower-triangular structure of (6). There is no loss of generality in this assumption, because  $\mathcal{T}_{22} = 0$  implies that  $\mathcal{T}_{22}$  is trivially block-lower-triangular, so we are free to transform  $\mathcal{T}$  as we transformed  $\mathcal{P}$  in Section III.



**Lemma 26.** Suppose  $\mathcal{T} \in \mathcal{RH}_\infty$  satisfies (43), Assumptions C1–C4, and the structural condition (6). If

$$\mathcal{Q}_{11} = \left[ \begin{array}{c|c} A_P & B_P \\ \hline C_P & 0 \end{array} \right]$$

then the unique solution to (44) is

$$[\mathcal{Q}_{21} \quad \mathcal{Q}_{22}] = \left[ \begin{array}{ccc|c} A + B_2 E_2 \bar{K}_1 & B_2 E_2 \bar{K}_2 & B_2 (E_1 C_P + E_2 \bar{K}_3) & 0 \\ 0 & A + L C_2 & 0 & -L \\ 0 & 0 & A_P & B_P E_1^\top \\ \hline \bar{K}_1 & \bar{K}_2 & \bar{K}_3 & 0 \end{array} \right] \quad (49)$$

Here the quantities  $\bar{K}_1$ ,  $\bar{K}_2$ , and  $\bar{K}_3$  are defined by

$$\begin{aligned} \bar{K}_1 &= [-R_{22}^{-1} (B_{22}^\top \Theta_X + S_{12}^\top) \quad J] & \bar{K}_2 &= [-R_{22}^{-1} (B_{22}^\top \Phi + S_{12}^\top) \quad J] \\ \bar{K}_3 &= -R_{22}^{-1} (B_{22}^\top \Gamma_X + R_{12}^\top C_P) \end{aligned}$$

where  $\Theta_X$ ,  $\Gamma_X$ , and  $\Phi$  are the unique solutions to the linear equations

$$\begin{aligned} A_J^\top \Theta_X + \Theta_X A_{11} + \tilde{X} A_{21} + J^\top S_{12}^\top + \mathcal{Q}_{21} &= 0 \\ A_J^\top \Gamma_X + \Gamma_X A_P + (\Theta_X B_{11} + \tilde{X} B_{21} + J^\top R_{21} + S_{21}) C_P &= 0 \\ A_J^\top \Phi + \Phi A_{11} + \tilde{X} A_{21} + J^\top S_{12}^\top + \mathcal{Q}_{21} + \Gamma_X B_P C_{11} &= 0 \end{aligned} \quad (50)$$

and  $Q, R, S$  are defined in (14), and  $\tilde{X}, Y, J, L$  are defined in (15).

**Proof.** First, we must find a joint realization for the blocks of  $\mathcal{T}_{11} + \mathcal{T}_{12} \mathcal{Q} \mathcal{T}_{21}$ . Since  $\mathcal{Q}_{11}$  is held fixed, we group it with  $\mathcal{T}_{11}$  as we did in the proof of Lemma 25 and obtain

$$\begin{aligned} \left[ \begin{array}{cc|cc} \mathcal{T}_{11} + \mathcal{T}_{12} E_1 \mathcal{Q}_{11} E_1^\top \mathcal{T}_{21} & \mathcal{T}_{12} E_2 \\ \mathcal{T}_{21} & 0 \end{array} \right] &= \left[ \begin{array}{ccc|cc} A & 0 & B_2 E_1 C_P & 0 & B_2 E_2 \\ 0 & A & 0 & B_1 & 0 \\ 0 & B_P E_1^\top C & A_P & B_P E_1^\top D_{21} & 0 \\ \hline C_1 & C_1 & D_{12} E_1 C_P & 0 & D_{12} E_2 \\ 0 & C_2 & 0 & D_{21} & 0 \end{array} \right] \\ &= \left[ \begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & 0 & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & 0 \end{array} \right] \end{aligned}$$

It is straightforward to check that Assumptions B1–B3 are satisfied for this augmented system. Now, we may apply Proposition 24. The result is that

$$[\mathcal{Q}_{21} \quad \mathcal{Q}_{22}]_{\text{opt}} = \left[ \begin{array}{c|c} \bar{A} + \bar{B}_2 \bar{K} & \bar{B}_2 \bar{K} \\ \hline 0 & \bar{A} + \bar{L} \bar{C}_2 \\ \hline \bar{K} & \bar{K} \end{array} \middle| \begin{array}{c} 0 \\ -\bar{L} \\ 0 \end{array} \right] \quad (51)$$

where  $(\bar{X}, \bar{K}) = \text{ARE}(\bar{A}, \bar{B}_2, \bar{C}_1, \bar{D}_{12})$  and  $(\bar{Y}, \bar{L}^\top) = \text{ARE}(\bar{A}^\top, \bar{C}_2^\top, \bar{B}_1^\top, \bar{D}_{21}^\top)$ . One can check that the stabilizing solution to the latter ARE is

$$\bar{Y} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \bar{L} = \begin{bmatrix} 0 \\ L \\ -B_P E_1^\top \end{bmatrix}$$

The former ARE is more complicated, however. Examining  $\bar{K} = -\bar{R}^{-1}(\bar{B}^\top \bar{X} + \bar{S}^\top)$ , we notice that due to all the zeros in  $\bar{B}$ , the only part of  $\bar{X}$  that affects the gain  $\bar{K}$  is the second sub-row of the first block-row. In other words, if

$$\bar{X} = \begin{bmatrix} \bar{X}_{11} & \bar{X}_{12} & \bar{X}_{13} \\ \bar{X}_{21} & \bar{X}_{22} & \bar{X}_{23} \\ \bar{X}_{31} & \bar{X}_{32} & \bar{X}_{33} \end{bmatrix}$$

then  $\bar{K}$  only depends on  $E_2^\top \bar{X}_{11}$ ,  $E_2^\top \bar{X}_{12}$ , and  $E_2^\top \bar{X}_{13}$ . Multiplying the ARE on the left by  $[E_2^\top \ 0 \ 0]$ , we can write a single equation that determines the quantities of interest.

$$\begin{aligned} A_{22}^\top [E_2^\top \bar{X}_{11} \ E_2^\top \bar{X}_{12} \ E_2^\top \bar{X}_{13}] + [E_2^\top \bar{X}_{11} \ E_2^\top \bar{X}_{12} \ E_2^\top \bar{X}_{13}] \begin{bmatrix} A & 0 & BE_1 C_P \\ 0 & A & 0 \\ 0 & B_P E_1^\top C & A_P \end{bmatrix} \\ - (E_2^\top \bar{X}_{11} E_2 B_{22} + S_{22}) R_{22}^{-1} (B_{22}^\top [E_2^\top \bar{X}_{11} \ E_2^\top \bar{X}_{12} \ E_2^\top \bar{X}_{13}] \\ + [E_2^\top S^\top \ E_2^\top S^\top \ R_{21} C_P]) + [E_2^\top Q \ E_2^\top Q \ S_{21} C_P] = 0 \end{aligned} \quad (52)$$

Right-multiplying by  $[E_2^\top \ 0 \ 0]^\top$  or by  $[0 \ E_2^\top \ 0]^\top$ , we obtain an equation of the form

$$A_{22}^\top \tilde{X} + \tilde{X} A_{22} - (\tilde{X} B_{22} + S_{22}) R_{22}^{-1} (\tilde{X} B_{22} + S_{22})^\top + Q_{22} = 0$$

and this is simply the control ARE corresponding to the 22 subsystem  $\dot{x}_2 = A_{22}x_2 + B_{22}u_2$ . Therefore, we conclude that  $E_2^\top \bar{X}_{11} E_2 = E_2^\top \bar{X}_{12} E_2 = \tilde{X}$ . The corresponding gain is  $J = -R_{22}^{-1}(B_{22}^\top \tilde{X} + S_{22}^\top)$ . Equation (52) simplifies to

$$\begin{aligned} A_J^\top [E_2^\top \bar{X}_{11} \ E_2^\top \bar{X}_{12} \ E_2^\top \bar{X}_{13}] + [E_2^\top \bar{X}_{11} \ E_2^\top \bar{X}_{12} \ E_2^\top \bar{X}_{13}] \begin{bmatrix} A & 0 & BE_1 C_P \\ 0 & A & 0 \\ 0 & B_P E_1^\top C & A_P \end{bmatrix} \\ + J^\top [E_2^\top S^\top \ E_2^\top S^\top \ R_{21} C_P] + [E_2^\top Q \ E_2^\top Q \ S_{21} C_P] = 0 \end{aligned} \quad (53)$$

Notice that (53) is linear in the  $\bar{X}$  terms. Assign the following names to the missing pieces

$$E_2^\top \bar{X}_{11} \triangleq [\Theta_X \ \tilde{X}] \quad E_2^\top \bar{X}_{12} \triangleq [\Phi \ \tilde{X}] \quad E_2^\top \bar{X}_{13} \triangleq \Gamma_X$$

Upon substituting these definitions into (53) and simplifying, we obtain (50). The associated control gain  $\bar{K}$  is given by  $\bar{K} = [\bar{K}_1 \ \bar{K}_2 \ \bar{K}_3]$ , where the individual components are defined by

$$\begin{aligned} \bar{K}_1 &= [-R_{22}^{-1} (B_{22}^\top \Theta_X + S_{12}^\top) \ J] \quad \bar{K}_2 = [-R_{22}^{-1} (B_{22}^\top \Phi + S_{12}^\top) \ J] \\ \bar{K}_3 &= -R_{22}^{-1} (B_{22}^\top \Gamma_X + R_{12}^\top C_P) \end{aligned}$$

Now, substitute  $\bar{K}$  and  $\bar{L}$  into (51). The result is a very large state-space realization, but it can be greatly reduced by eliminating uncontrollable and unobservable states. The result is (49). This reduction is not surprising, because we solved a model-matching problem in which the joint realization for the three blocks had a zero as the fourth block. ■

We may solve (45) in a manner analogous to how we solved (44). Namely, we can provide a formula for the optimal  $\mathcal{Q}_{11}$  and  $\mathcal{Q}_{21}$  as functions of  $\mathcal{Q}_{22}$ . The result follows directly from Lemma 26 after we make a change of variables.

**Lemma 27.** Suppose  $\mathcal{T} \in \mathcal{RH}_\infty$  satisfies (43), Assumptions C1–C4, and the structural condition (6). If

$$\mathcal{Q}_{22} = \left[ \begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & 0 \end{array} \right]$$

then the unique solution to (45) is

$$\begin{bmatrix} \mathcal{Q}_{11} \\ \mathcal{Q}_{21} \end{bmatrix} = \left[ \begin{array}{ccc|c} A + \bar{L}_1 E_1^\top C_2 & 0 & 0 & \bar{L}_1 \\ \bar{L}_2 E_1^\top C_2 & A_K & 0 & \bar{L}_2 \\ (\bar{L}_3 E_1^\top + B_Q E_2^\top) C_2 & 0 & A_Q & \bar{L}_3 \\ \hline 0 & -K & E_2 C_Q & 0 \end{array} \right] \quad (54)$$

The quantities  $\bar{L}_1$ ,  $\bar{L}_2$ , and  $\bar{L}_3$  are defined by

$$\begin{aligned} \bar{L}_1 &= \begin{bmatrix} M \\ -(\Theta_Y C_{11}^\top + U_{12}^\top) V_{11}^{-1} \end{bmatrix} & \bar{L}_2 &= \begin{bmatrix} M \\ -(\Psi C_{11}^\top + U_{12}^\top) V_{11}^{-1} \end{bmatrix} \\ \bar{L}_3 &= -(\Gamma_Y C_{11}^\top + B_Q V_{21}) V_{11}^{-1} \end{aligned}$$

where  $\Theta_Y$ ,  $\Gamma_Y$ , and  $\Psi$  are the unique solutions to the linear equations

$$\begin{aligned} A_{22} \Theta_Y + \Theta_Y A_M^\top + (A_{21} \tilde{Y} + U_{12}^\top M^\top + W_{21}) &= 0 \\ A_Q \Gamma_Y + \Gamma_Y A_M^\top + B_Q (V_{21} M^\top + U_{21} + C_{21} \tilde{Y} + C_{22} \Theta_Y) &= 0 \\ A_{22} \Psi + \Psi A_M^\top + (A_{21} \tilde{Y} + U_{12}^\top M^\top + W_{21} + B_{22} C_Q \Gamma_Y) &= 0 \end{aligned} \quad (55)$$

and  $W, V, U$  are defined in (14), and  $X, \tilde{Y}, K, M$  are defined in (15).

A key observation that greatly simplifies the extent to which the two optimization problems (44) and (45) are coupled is to notice that the  $\mathcal{Q}_{ii}$  have simple state-space representations.

**Remark 28.** If we isolate  $\mathcal{Q}_{11}$  from (54) by multiplying on the left by  $E_1^\top$ , we have  $E_1^\top E_2 C_Q = 0$ , and the state-space realization can be significantly reduced. A similar procedure can be applied to (49) to isolate  $\mathcal{Q}_{22}$ . After simplification, we obtain

$$\mathcal{Q}_{11} = \left[ \begin{array}{cc|c} A_K & \bar{L}_2 C_{11} & \bar{L}_2 \\ 0 & A_M & M \\ \hline -E_1^\top K & 0 & 0 \end{array} \right] \quad \mathcal{Q}_{22} = \left[ \begin{array}{cc|c} A_J & B_{22} \bar{K}_2 & 0 \\ 0 & A_L & -L E_2 \\ \hline J & \bar{K}_2 & 0 \end{array} \right] \quad (56)$$

Consequently, we come to a surprising conclusion, that the modes of  $\mathcal{Q}_{22}$  do not depend on  $\mathcal{Q}_{11}$ . The entirety of  $\mathcal{Q}_{22}$ 's dependence on  $\mathcal{Q}_{11}$  is encapsulated in  $\bar{K}_2$ . Similarly,  $\mathcal{Q}_{11}$  only depends on  $\mathcal{Q}_{22}$  through  $\bar{L}_2$ , and its modes are independent as well.

Lemmas 26 and 27 provide solutions to (44) and (45). It remains to enforce the constraint that these solutions be consistent.

**Theorem 29.** Suppose  $\mathcal{T} \in \mathcal{RH}_\infty$  satisfies (43), Assumptions C1–C4, and the structural condition (6). The solution to the two-player model-matching problem (22) is

$$\mathcal{Q}_{\text{opt}} = \left[ \begin{array}{ccc|c} A_K & -\hat{L} C_2 & 0 & -\hat{L} \\ 0 & \hat{A} & -B_2 \hat{K} & \hat{L} \\ 0 & 0 & A_L & L \\ \hline K & \hat{K} & -\hat{K} & 0 \end{array} \right] \quad (57)$$

where  $K, L, \hat{K}, \hat{L}$  are defined in (15)–(18) and  $A_K, A_L, \hat{A}$  are defined in (26).

**Proof.** First, we take care of existence and uniqueness. From Lemma 13, (22) has a unique rational solution. By Lemma 25, there must exist a unique simultaneous rational solution to (44) and (45), and it must be given by the expressions in Lemmas 26 and 27.

We enforce that (54) and (49) match by first enforcing that  $\mathcal{Q}_{11}$  agrees. Substituting (56) into Lemma 26 gives

$$\left[ \begin{array}{c|c} A_P & B_P \\ \hline C_P & 0 \end{array} \right] = \left[ \begin{array}{cc|c} A_K & \bar{L}_2 C_{11} & \bar{L}_2 \\ 0 & A_M & M \\ \hline -E_1^\top K & 0 & 0 \end{array} \right]$$

The resulting system of equations (50) then becomes

$$A_J^\top \Theta_X + \Theta_X A_{11} + (\tilde{X} A_{21} + J^\top S_{12}^\top + Q_{21}) = 0 \quad (58)$$

$$A_J^\top \Gamma_X + \Gamma_X \begin{bmatrix} A_K & \bar{L}_2 C_{11} \\ 0 & A_M \end{bmatrix} + (\Theta_X B_{11} + \tilde{X} B_{21} + J^\top R_{21} + S_{21}) \begin{bmatrix} -E_1^\top K & 0 \end{bmatrix} = 0 \quad (59)$$

$$A_J^\top \Phi + \Phi A_{11} + (\tilde{X} A_{21} + J^\top S_{12}^\top + Q_{21} + \Gamma_X \begin{bmatrix} \bar{L}_2 \\ M \end{bmatrix} C_{11}) = 0 \quad (60)$$

Note that (58) is independent of the other two equations, and uniquely determines  $\Theta_X$  because it is a Sylvester equation and both  $A_J$  and  $A_{11}$  are stable. It can be verified by direct substitution that the solution to (59) is given by

$$\Gamma_X = \begin{bmatrix} \Theta_X - X_{21} & \tilde{X} - X_{22} & \Phi - \Theta_X \end{bmatrix}$$

Substituting  $\Gamma_X$  into (60) and simplifying, we obtain an equation for  $\Phi$

$$A_J^\top \Phi + \Phi A_{11} + (\tilde{X} A_{21} + J^\top S_{12}^\top + Q_{21}) + [\Phi - X_{21} \quad \tilde{X} - X_{22}] \bar{L}_2 C_{11} = 0$$

Note that this equation also contains  $\bar{L}_2$ , which only depends on  $\Psi$ . We can make the dependence on  $\Phi$  and  $\Psi$  more explicit by expanding  $\bar{L}_2$  and collecting terms. Doing so, we obtain (16).

Next, we follow a similar procedure to enforce that  $\mathcal{Q}_{22}$  matches. Substituting equation (56) into Lemma 27, we find

$$\Gamma_Y = \begin{bmatrix} \Psi - \Theta_Y \\ \tilde{Y} - Y_{11} \\ \Theta_Y - Y_{21} \end{bmatrix}$$

and obtain an equation for  $\Psi$  that contains  $\bar{K}_2$

$$A_{22} \Psi + \Psi A_M^\top + (A_{21} \tilde{Y} + U_{12}^\top M^\top + W_{21}) + B_{22} \bar{K}_2 \begin{bmatrix} \tilde{Y} - Y_{11} \\ \Psi - Y_{21} \end{bmatrix} = 0$$

Substituting the definition for  $\bar{K}_2$ , we obtain (17). Finally, we return to (49) and substitute in the expression we found for  $\Gamma_X$  and our expression for  $C_P$ . After some simplification,  $\Theta_X$  cancels and upon defining  $\hat{L} \triangleq \bar{L}_2 E_1^\top$  and  $\hat{K} \triangleq E_2 \bar{K}_2$ , we obtain (57). Using a similar procedure substituting  $\Gamma_Y$  into (54), we also obtain (57), which confirms that both (44) and (45) have been simultaneously solved. By Lemma 25,  $\mathcal{Q}_{\text{opt}}$  is the solution to the two-player model-matching problem.

The existence and uniqueness properties verified at the beginning of the proof imply that there exists unique  $\Phi$  and  $\Psi$  which simultaneously satisfy (16)–(17). ■

## VII Summary

In this article, we studied the class of two-player output feedback problems with a nested information pattern. We began by considering the problem of realization of block lower-triangular systems, and showed that, surprisingly, no additional states were required when imposing the constraint that the state-space matrices for such systems be triangular.

We then presented an extensive state-space treatment of stabilization, giving necessary and sufficient conditions for the existence of a structured stabilizing controller. Here also there are surprises. In the centralized case, every plant may be stabilized in the input-output sense. However, there are plants for which there is no decentralized stabilizing controller. We then address the construction of a Youla-like parameterization of all such controllers. Instead of an input-output formulation, we give a state-space parameterization of all stabilizing controllers. This gives an immediately applicable and computable result, which we subsequently use for controller synthesis.

The main result of this paper is the explicit state-space formulae for the optimal  $\mathcal{H}_2$  controller for the two-player output feedback problem. Just as in the centralized case, one is interested in more than just a formula. In the centralized case, it is a celebrated and widely-generalized result that the controller is a composition of an optimal state-feedback gain with a Kalman filter estimator. Our approach generalizes both the centralized formulae and this separation structure to the two-player decentralized case. We show that the  $\mathcal{H}_2$ -optimal structured controller has twice the state dimension of the plant, and we give intuitive interpretations for the states of the controller as steady-state Kalman filters. The player with more information must duplicate the estimator of the player with less information. This has the simple anthropomorphic interpretation that Player 2 is correcting mistakes made by Player 1. The controller also has a certainty equivalence property, so that if the state-estimates are replaced by the true states then the control action is precisely that which would be taken by the optimal centralized state-feedback controller.

Both the state-space dimension and separation structure of the optimal controller were previously unknown. While these results show that the optimal controller for this problem has an extremely simple state-space structure, not all such decentralized problems exhibit such pleasant behavior. One example is the two-player fully-decentralized state-feedback problem, where even though the optimal controller is known to be linear and rational it is shown in [10] that the number of states of the controller may grow quadratically with state-dimension.

The formulae that we give for the optimal controller are simply computable, requiring the solution of four standard algebraic Riccati equations, two that have the same dimension as the plant and two with a smaller dimension. In addition, one must solve two linear matrix equations. All of these computations are simple and have readily available existing code.

We analyze the cost of the optimal controller and give an expression that explicitly shows the cost incurred by decentralization. The excess cost is equal to the  $\mathcal{H}_2$  norm of a specific auxiliary system.

While there is as yet no complete state-space theory for decentralized control, in this work we provide solutions to a prototypical class of problems which exemplify many of the features found in more general problems. It remains a fundamental and important problem to fully understand the separation structure of optimal decentralized controllers in the general case. While we solve the two-player triangular case, we hope that the solution gives some insight and possible hints regarding the currently unknown structure of the optimal controllers for more general architectures.

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